STUDIES IN HIGHER DIMENSIONAL AND HIGHER ORDER GRAVITY

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CERTIFICATE

It is certified that the work contained in the thesis entitled STUDIES IN HIGHER DIMENSIONAL AND HIGHER ORDER GRAVITY, by BIPLAB BHAWAL, has been carried out under our supervision and that this work has not been submitted elsewhere for a degree.

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SYNOPSIS

In the last few years the search for a consistent theory of quantum gravity and the quest for a unification of gravity with other forces have both led to a spurt of interest in theories with extra spatial dimensions incorporated in them, e.g., Kaluza-Klein theory, Superstring, Supergravity etc. Another subtle aim of studying higher dimensions is to explain why the observed universe is so specific to choose 1+3 dimensions although always there exists a greater generality in the statements and mathematics describing the laws of nature.

It can be shown that the Lagrangian of a higher dimensional theory may naturally incorporate some additional higher order terms. These terms also appear in the low frequency limit of the superstring theory. In the absence of the knowledge of the complete theory of quantum gravity, from time to time, attempts have been made to gain further insight by studying models which include only leading order corrections.

This thesis aims to study various physical processes associated with some models of such theories with a view to isolating signatures of higher dimensional and higher order terms. We essentially concentrate our study on higher dimensional Einstein action modified by the Gauss–Bonnet combination, the leading second order correction. It highlights especially two distinctive features of higher dimensional gravity — the nonexistence of bound states in central mass problems and the semiclassical decay of the ground state. Several related works have been done with the aim of probing the geometry and realising the essential differences between models of such theories and those in ordinary four dimensional general relativity.

In chapter I, we first set the motivation for studying higher dimensional theories and present a chronology and brief review of the developments in this line. Next we go on to introduce Lovelock gravity that naturally incorporates higher order terms in the Lagrangian of higher dimensional theories and mention the implication of such terms with

a special reference to superstring theory.

One can show that all such terms can be identified with the dimensionally extended Euler forms corresponding to each of the even number of dimensions. An analysis (based on the existing literature) of such terms in the language of differential forms has also been presented. The last section enlists and summarizes some works already done on some models of these theories.

In chapter II, we concentrate our study on a simple model of such a theory— a static spherically symmetric solution— the Boulware Deser Black Hole (BDBH). We specifically studied two cases first— the geodesic motion and the Hawking radiation.

In the former case, we compared the results with those in higher dimensional Schwarzschild spacetime solution of the Einstein–Hilbert Lagrangian. An interesting result of this study is the nonexistence of any stable bound orbit in higher dimensional black hole spacetime. This can be thought to be a general relativistic analogue of the consequences of the Bertrand's theorem in Newtonian Mechanics. This is an important distinctive feature of the higher dimensional theory. We also observed that the presence of higher order terms in the action does not significantly affect the nature of geodesic orbits, except changing the position of the horizon and the maximum point of the effective potential. Associated with this section is Appendix A where we present generalized expressions for the deflection of null ray in higher dimensional black hole spacetime.

Our study of the Hawking radiation from BDBH is based on the solution of the scalar field in such a spacetime. One may see that the techniques used in the four dimensional case can be extended to any higher dimensional static spherically symmetric black hole solution in a straight forward manner. The case of five dimensional BDBH is unique in the sense that only in that case one can identify the cosmic censorship hypothesis with the third law of black hole thermodynamics. The last section contains some comments on the back reaction problem. Since the detailed back reaction problem is, in general, notoriously difficult to untangle, we attempt to make some guesses on this and the singularity formation

process.

Chapters III, IV and V concentrate on another very important unique feature of higher dimensional gravity— the semiclassical decay of the ground state. Such a process can never occur in four dimensional general relativity because the positive energy theorem ensures the uniqueness of the ground state (four dimensional flat Minkowski spacetime).

Chapter III is a review of the works done in this line. We first introduce the concept and salient features of the semiclassical decay process of false vacuum in ordinary quantum field theory (without gravitation). Then we summarize how the situation becomes highly nontrivial when one applies these concepts to the gravitational field.

We describe important steps in Witten's proof of the positive energy theorem based on spinor algebra. This proof cannot be fully generalized to higher dimensional gravity. The difficulties arise when one considers multiply connected spacetimes like $M^4 \times S^1$. We describe how Witten proved the semiclassical instability of the flat spacetime of such a topology. The spacetime into which the ground state decays is known as Witten bubble.

In chapter IV, we study massless scalar waves in the Witten bubble background. Such a study is essential for understanding the classical properties and the evolution behaviour of such a spacetime. We could write the timelike and angular parts of the separated Klein Gordon equation in terms of hyperbolic harmonics characterized by the generalized frequency ω. The radial equation is cast into the Schrödinger form. The coordinate transformation used for this purpose has a special meaning which has been discussed in Appendix C. The above mathematical formulation is applied to study the scattering problem, the bound states and the corresponding stability criteria. The results confirm the concept of the bubble wall as a perfectly reflecting expanding sphere. It has been found that bound states as well as quasi-normal modes are absent. It has been shown that the bubble spacetime is stable with respect to any arbitrary scalar perturbation. In Appendix B, we also present an alternative scalar wave solution in this spacetime.

In chapter V, we study the semiclassical decay of the $M^4 \times S^1$ ground state in higher

order gravity. We also address the question of the validity of the positive energy theorem in this theory. We obtained two instanton solutions representing the decay. Correspondingly, two alternative Lorentzian spacetimes (into which the ground state decays) have been obtained. The first solution, in the limit of vanishing coupling constant for the Gauss-Bonnet combination, approaches the Witten bubble solution, whereas the second solution is an entirely new one. Both solutions have similar qualitative features as those of the Witten Bubble.

Chapter VI, the last chapter, mentions some important works done by other people and highlights some prospects as well as problems in higher order gravity. It summarizes and reviews our work in this broader perspective, and proposes logical extensions and improvements thereof.

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STUDIES IN

HIGHER DIMENSIONAL AND HIGHER ORDER GRAVITY

Contents

LIST OF FIGURES

* Boulware Deser Black Hole

NOTATION

(1) **Indices** : Latin indices represent all the spatial coordinates ($1, 2, 3, \ldots$ etc. upto $D-1$ for D dimensional spacetime), whereas Greek indices denote all coordinates including both temporal (0) and spatial ones ($1,2,3, \ldots$ etc.). While using the language of differential forms, we will represent the coordinates of the orthonormal basis by the capital letter Latin indices ($A, B, C, \ldots$ etc.).

(2) **Signature** : We choose signature $(- \ + \ + \cdots +)$ for the spacetime metric. The corresponding rule is that timelike intervals are imaginary and spacelike ones are real.

(3) **Summation Convention** : Einstein Summation Convention is always followed, i.e. repeated index will be summed over all the values it may take.

(4) **Units** : Depending on the convenience and judging the relevance of the actual values to our discussion, we shall frequently set one or more of the following four fundamental constants to unity— c, the speed of light, G, the gravitational constant, k_B, the Boltzmann constant, $\hbar$, the Planck constant[$/(2\pi)$]. We indicate our choice in every case in our discussion.

Chapter I

INTRODUCTION

The fact that General Relativity (GR) is the correct classical theory of gravity has been well established by sufficient experimental evidence. It is believed that this theory is the low frequency limit of a quantum theory of gravity which is yet to be developed. In the last few years the search for a consistent quantum theory of gravity and the quest for a unification of gravity with other forces have both led to a renewed interest in theories with extra spatial dimensions incorporated in them.

In this chapter, we first establish the motivation for studying higher dimensional Gravity and present a brief chronology and review of the works done in this line. We then go on to introduce higher order terms that may be naturally incorporated in the Lagrangian of such theories. In the third section, we present an analysis based on the existing literature of these higher order terms in the language of Differential Forms. The last section is a brief review of the features of some solutions of these theories. This chapter, as a whole, will provide the essential groundwork for the topics to be discussed in the rest of the chapters.

I.(A) Higher Dimensional Gravity

When Einstein first talked of a new kind of Physics in a 4-dimensional manifold unifying space and time, a large cross section of the society including physicists as well as nonspecialists raised eyebrows either in disbelief or in bewilderment. The 3-dimensional realm of the Newtonian Mechanics was an adequate framework to common human mind for studying phenomena encountered in its daily life. The special theory of relativity was revolutionary in the sense that it first disabused people of the idea that Physics is just the study of objects in their familiar 3-dimensional world where time acts just as a parameter.

A beautiful equation like $E = mc^2$ was an outcome of this kind of Physics in 4-manifold

and could never be conceived through the Newtonian Mechanics. The real implication of this formula could be convincingly verified in different processes in nuclear and particle Physics.

Once this breakthrough was achieved and the psychological stumbling blocks got removed, there was no bar in extending the limit of imagination to higher ($D > 4$) dimensional manifolds by incorporating extra spatial dimensions *. In fact, in any of the basic laws of nature (e.g. Newton's laws of motion, the Lagrangian and Hamiltonian formalisms, principles of the special relativity, Principle of equivalence, Principle of general covariance, the geodesic principle, the quantum mechanics principles), neither the statement of the principle nor the mathematical machineries were ever restricted to three dimensions. The general hope was that such an exercise might as well be successful in giving us some distinctive results which can again be verified by either experiments or the observation of the universe.

T. Kaluza(1921) and O. Klein(1926) [see Appelquist et al.,1987, for the original papers and English translation thereof] suggested that gravitation and electromagnetism could be unified in a theory of 5-dimensional Riemannian Geometry. Over the years, however, such an idea became decrepit because of its failure to provide some unique verifiable prediction; instead, it yielded some unphysical results when combined with quantum theory [Bailin and love,1987].

However, with the growing importance of gauge invariance as a major guiding principle for the formulation of physical laws, the urgency of unifying gauge fields with General Relativity (GR) began to be strongly felt. So, in the 70's this approach was resurrected again with much more vigour [Scherk and Schwarz, 1975, Cho, 1975, Cremmer and Scherk, 1976].

In contrast to the classical literature, this modern approach [Witten, 1981b, Salam

* Theories with additional time-like dimensions appear to be plagued with ghosts, see e.g. Duff, Nilsson, and Pope (1986)

and Stathdee, 1982; For a concise review, see Bailin and Love, 1987] established the idea that the extra dimensions should be regarded as true, physical dimensions on par with the four observed dimensions and not be treated as a mere mathematical device. In this line of attack the gauge invariance assumes the same status as spacetime invariance and internal symmetries originate in the spacetime symmetries associated with the extra dimensions. In this framework, therefore, it is essential that at every stage in the derivation of the effective 4-dimensional field theory, one maintains consistency with the higher dimensional field equations.

The sizes of the extra dimensions are free parameters of the model, ones that are not determined even when we specify all parameters of the Lagrangian. However, establishing a relationship among the coupling constants, gravitational constant G and the size of the extra dimensions (each coupling is given by the ratio of $2\pi(16\pi G)^{1/2}$ to an arbitrary root-mean-square circumference), Weinberg (1983) and Candelas and Weinberg (1984) suggested that the size of the extra dimensions are at most a few orders of magnitude greater than the Planck length ($\sim 10^{-33}$cm.). Since the present day accelerators can probe matter at 10^{-16} cm. only, resolving the extra dimensions at currently available energies is out of question. But this may not have been always so. If we go back in time, according to the standard cosmological model, there must have been a time when the visible universe was of a comparable size to that of the internal space. The present difference between the four observed dimensions and the extra microscopic ones could arise from a spontaneous breakdown of the vacuum symmetry i.e.'spontaneous compactification' of the extra dimensions.

The idea of extra hidden dimensions also stimulated much work in supersymmetry theory. This idea permits a simple derivation of the $SO(8)$ supergravity Lagrangian by an appropriate dimensional reduction of the $N = 1$ supergravity Lagrangian in eleven dimensions [Cremmer and Julia, 1979].

The foundation of the superstring theory has also been built on spacetime of more

than four dimensions. In this theory, the dimension of the Minkowskian spacetime in which the first quantization can be perturbatively done turns out to be $D = 10$, while for the old bosonic string, this same number is $D = 26$ [Green et al., 1987].

The fact that higher dimensions naturally arise in different unification theories involving gravitation provides the main motivation for studying higher dimensional gravity. However, we should remember that there remains a more subtle aim of such a study. This direction of investigation concentrates on attempting to explain why the observed space is so specific to choose three dimensions, although almost always there exists a greater generality in the statements and mathematics describing the laws of nature. Such a question should also be addressed to all the unification schemes since those also involve the 'specificity' of choosing a particular number of dimensions.

I.(B) Higher order Terms

Einstein's gravitational tensor (together with the cosmological term) was, in any dimension, the only symmetric and conserved tensor depending only on the metric and its first and second derivatives, with a linear dependence on the latter. He was able to deduce the simple Ricci scalar Lagrangian only by making certain simplifying assumptions [Einstein, 1916]. The gravity Lagrangian could, in fact, contain an arbitrary number of terms consisting of the invariants which can be constructed from powers of the Riemann curvature tensor. It is hard to argue on experimental ground that such additional terms should not be present since, in all practical situations, the curvature is very small. It was Weyl(1919) who first introduced such terms in his affine theory which claimed to unify gravity and electromagnetism.

It is false to assume that adding a higher order correction term with a small coefficient will only perturb the original theory. The presence of an unconstrained higher order term, no matter how small it may naively appear, may make the new theory dramatically different from the original.

The classical Einstein theory should be the low energy limit of a quantum theory of

gravity. It was suggested that the action for a quantum theory of gravity should contain some nonminimal functionals of the metric tensor which involve more than two derivatives. The action gets modified by higher order interactions in any attempt to perturbatively quantize gravity as a field theory [Grisaru *et al*, 1976; Deser, Kay, Stelle,1977; Goroff and Sagnotti, 1986]. Gravitational actions which include terms quadratic in curvature tensor are renormalizable [Stelle, 1977; Birrel and Davies, 1982].

It is hoped that the full low energy theory will solve the problem of singularities in GR. However, in the absence of the knowledge of the details of such a theory, attempts have been made to gain further insight by studying models which include only the leading order corrections.

From time to time people studied different objects and issues arising in ordinary GR in the context of four dimensional theories involving higher powers of the curvature tensor (e.g., $R + R^2$ theory etc.). The associated field equations are typically of fourth order in the derivatives and are exceedingly nonlinear [For a review of such theories, see Boulware, Strominger, Tomboulis, 1984].

In cosmology, such terms were introduced first by Starobinsky (1980,1983) with an aim to avoid the initial singularity. It was found that such models may lead to inflationary expansion driven only by gravity.

Recent developments in the superstring approach to the unification of all forces have also provided concrete suggestions for higher order corrections to the Einstein action. In their field theoretic limit [Scherk and Schwarz, 1974], string theories give rise to effective models of gravity in higher dimensions which involve higher powers of Riemann curvature. Of these, the quadratic term is of particular importance because it is the leading one and can affect the gravitational excitation spectrum near the flat space.

However, if like the string itself, its slope expansion is to be ghost free, the quadratic term, if any, must be the Gauss-Bonnet (GB) combination [Zwiebach, 1985]

$$(\mathrm{GB}) \equiv R_{\alpha\beta\gamma\delta}{}^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2. \qquad (I.1)$$

Such an addition would not modify the propagator because now if one expands the action about Minkowski space, the terms quadratic in the gravitational field combine to a total derivative and integrate to zero. In four dimensions the combination multiplied by $\sqrt{-g}$ is a total derivative and proportional to the Euler topological invariant.

Therefore, the Gauss-Bonnet combination can act as the leading correction to Einstein theory in the low frequency effective field theory of the string only for dimensions greater than four.

The presence of these higher order terms can also be understood from the point of view of Lovelock's theorem. It is interesting to note that Lovelock [1971,72] was also led to this action by a different route.

Within the realm of classical gravity, Lovelock tried to obtain the most general second rank tensor in arbitrary dimensions, which is (i) symmetric, (ii) depends on the metric and its derivatives upto second order and (iii) divergence free. He relaxed the requirement that the tensor be linear in second derivatives of the metric. However, it turns out that in 4-dimensions, this follows naturally from the above assumptions and nonlinear terms arise only in higher dimensions.

The Lovelock theorem states that in D-dimensions, the number of such independent tensors is $m = D/2$ for D even and $m = (D+1)/2$ for D odd. The most general metric Lagrangian is given by a finite sum of the dimensionally extended Euler densities (to be explained in detail in section I.3)

$$A_\mu{}^\nu = \sum_{p=1}^{m-1} a_p A^{(p)}{}_\mu{}^\nu + a\delta_\mu{}^\nu \qquad (I.2a)$$

$$A^{(p)}{}_\mu{}^\nu = \delta^{\nu\nu_1\cdots\nu_{2p}}_{\mu\mu_1\cdots\mu_{2p}} R_{\nu_1\nu_2}{}^{\mu_1\mu_2}\cdots R_{\nu_{2p-1}\nu_{2p}}{}^{\mu_{2p-1}\mu_{2p}} \qquad (I.2b)$$

where $a_p(p = 1, \cdots, m-1)$ and 'a' are arbitrary constants. The generalised Kronecker delta symbol is given as

$$\delta^{\nu_1\cdots\nu_N}_{\mu_1\cdots\mu_N} = \delta^{[\nu_1}_{[\mu_1}\delta^{\nu_2}_{\mu_2}\cdots\delta^{\nu_N]}_{\mu_N]}. \qquad (I.2c)$$

In 10 dimensions, for example, we would expect terms in the action of up to quartic order in the curvature. The quantity 'a' is equivalent to the cosmological constant. The lowest order term ($p = 1$) in the summation is identical with the Einstein tensor, whereas the ($p = 2$) term corresponds to the leading quadratic curvature correction or the Gauss-Bonnet combination.

Again, the field equations have the anomalous property that in $D > 4$, the tensor $A_\mu{}^\nu$ is nonlinear in the second order derivatives of $g_{\mu\nu}$ and differs from Einstein tensor only if the spacetime has more than four dimensions. Therefore, it yields the most natural generalisation of GR in higher dimensional spacetimes.

A misnomer : Before we end this section, we would like to point out that the Lovelock gravity is frequently referred to in the literature as a higher derivative theory. It is a misnomer to call such theories to be 'higher derivative' ones since the Lagrangian, just like the Einstein-Hilbert gravity, does not contain more than the second derivative of the metric. 'Higher Order Gravity' is an appropriate name for such theories.

I.(C) Higher Order Lagrangian In The Language Of Differential Forms

In this section we present an analysis of higher order terms arising in the Lagrangian of higher dimensional gravity by making use of the calculus of differential forms. This presentation is based on the existing literature on this subject [Zumino, 1986; Teitelboim and Zanelli, 1987]. Our aim is to systematize the whole procedure and present the same in a coherent way.

We consider a D-dimensional spacetime with a metric g of signature $(- + + \cdots +)$. Let e^A, $A = 1, \ldots D$ denote the orthogonal coframe (vielbein 1-forms).

$$g = \eta_{AB}\, e^A \otimes e^B \qquad \eta \equiv \mathrm{diag}(-1, 1, \ldots, 1) \tag{I.3}$$

Let us introduce the differential forms

$$\epsilon_{A_1\cdots A_m} = \frac{1}{(D-m)!} \epsilon_{A_1\cdots A_m A_{m+1}\cdots A_D}\ e^{A_{m+1}} \wedge \cdots \wedge e^{A_N}, \tag{I.4}$$

where $\epsilon_{A_1\cdots A_D}$ is totally antisymmetric with $\epsilon_{1\cdots D}= 1$. They satisfy

$$\begin{aligned} e^B \wedge \epsilon_{A_1\cdots A_m} &= \delta^B_{A_M}\ \epsilon_{A_1\cdots A_{m-1}} - \delta^B_{A_{m-1}}\ \epsilon_{A_1\cdots A_{m-2}A_m} \\ &+ \cdots + (-1)^{m-1}\ \epsilon_{A_2\cdots A_m} \end{aligned} \tag{I.5}$$

Let $\omega^A{}_B$ be a spin connection one-form compatible with the metric g. The torsion and curvature 2-forms are defined respectively as

$$T^A = de^A + \omega^A{}_B \wedge e^B = \mathcal{D}e^A \tag{I.6}$$

$$R^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B = \frac{1}{2} R^A{}_{BCD} e^C \wedge e^D = \frac{1}{2} R^A{}_{B\mu\nu} dx^\mu dx^\nu \tag{I.7}$$

$$\text{Also,} \quad R_{AB} = -R_{BA} \tag{I.8}$$

From now onwards, we stop indicating explicitly the wedge product sign.

The torsion and curvature 2-forms satisfy the Bianchi identities

$$dT^A + \omega^A{}_B T^B = R^A{}_B e^B = \mathcal{D}T^A \tag{I.9}$$

$$(dR + \omega R - R\omega)^A{}_B = (\mathcal{D}R)^A{}_B = 0 \tag{I.10}$$

A small variation $\delta\omega$ of the connection form induces a variation of R given by

$$\delta R = \mathcal{D}\delta\omega = d\delta\omega + \omega\delta\omega + \delta\omega\ \ \omega \tag{I.11}$$

The Lagrangian can be considered to be a linear combination of D-forms (in D dimensions) which, integrated over the manifold, gives the action. A particularly interesting class of D-forms invariant under local lorentz transformation are given by

$$L_{K,D-2K} = R^{A_1 B_1} R^{A_2 B_2} \cdots R^{A_K B_K} e^{A_{2K+1}} \cdots e^{A_D}\ \epsilon_{A_1 B_1 \cdots A_K B_K A_{2K+1} \cdots A_D}, \tag{I.12}$$

where $0 \leq K \leq D/2$. Therefore, in a given dimension, the number of Lagrangians considered here is finite.

Rewriting this expression in terms of the 4-index Riemann curvature tensor,

$$\begin{aligned}L_{K,D-2K} =&\frac{1}{2^K} R^{A_1 B_1}{}_{M_1 N_1} \cdots R^{A_K B_K}{}_{M_K N_K} e^{M_1} e^{N_1} \cdots e^{M_K} e^{N_K} e^{A_{2K+1}} \\ & \cdots e^{A_D} \in_{A_1 B_1 \cdots A_K B_K A_{2K+1} \cdots A_D} \\ =&\frac{(D-2K)!}{2^K} e R^{A_1 B_1}{}_{M_1 N_1} \cdots R^{A_K B_K}{}_{M_K N_K} \delta^{M_1 N_1 \cdots M_K N_K}_{A_1 B_1 \cdots A_K B_K} d^D x \qquad (I.13)\end{aligned}$$

where $e = \det e^A{}_\mu$.

When $D(=2E)$ is even, the particular $2E$-form corresponding to $K = E$ will be given by

$$L_{E,0} = R^{A_1 B_1} \cdots R^{A_E B_E} \in_{A_1 B_1 \cdots A_E B_E} \qquad (I.14)$$

This is proportional to the Euler invariant. For obvious reasons, Euler invariants corresponding to odd dimensions do not exist at all. Now, rewriting Eq.(I.15) in terms of 4-index Riemann curvature tensor,

$$L_{E,0} = \frac{1}{2^E} e R^{A_1 B_1}{}_{M_1 N_1} \cdots R^{A_E B_E}{}_{M_E N_E} \delta^{M_1 N_1 \cdots M_E N_E}_{A_1 B_1 \cdots A_E B_E} d^{2E} x \qquad (I.15)$$

So, barring a multiplying constant, both $L_{K,D-2K}$ and $L_{E,0}$ will generate the same expression, if $K = E$, except that the indices will run over different number of dimensions. In the first case, number of dimensions may be any $D > 2K$. In the latter case, it is equal to $2K$.

In essence, all the D-forms $L_{K,D-2K}$ can be interpreted as the extension to a higher number of dimensions D of the Euler number $L_{K,0}$ corresponding to dimensions $2K$. We may, therefore, represent the total Lagrangian as a linear combination of dimensionally extended Euler invariants :

$$\mathcal{L} = \sum_{K=0}^{p} \lambda_K L_K, \qquad \text{where} \quad p \leq p_{\max} \qquad (I.16a)$$

λ_K are constants

$$\text{and} \quad p_{\max} = \begin{cases} D/2 & \text{when } D \text{ is even} \\ (D-1)/2 & \text{when } D \text{ is odd.} \end{cases} \qquad (I.16b)$$

$$
\begin{aligned}
L_K &\equiv L_{K,0} \\
&= R^{A_1B_1}\dots R^{A_KB_K} \in_{A_1B_1\cdots A_KB_K}, \qquad 2K \leq D.
\end{aligned} \tag{I.17}
$$

In particular, we have

$$
\begin{aligned}
&L_0 = \in = \text{volume form}, \\
&L_1 = R^{AB} \in_{AB} = R \in, \\
&L_2 = R^{AB}R^{CD} \in_{ABCD} = (R_{ABCD}R^{ABCD} - 4R_{AB}R^{AB} + R^2) \in \\
&L_3 = R^{AB}R^{CD}R^{EF} \in_{ABCDEF} \\
&\quad \dots \text{etc.}
\end{aligned} \tag{I.18}
$$

So, the first term in L gives a cosmological constant, the second is proportional to the Einstein-Hilbert Lagrangian, the third is the Gauss-Bonnet combination. In $D = 2K$ dimensions, L_K is proportional to the Euler form which gives rise to the Euler (characteristic) class. In that case, there is no contribution to the field equations, since the entire expression ($\sqrt{-g}L_K$) is a total derivative. This will have contribution only in a theory with $D > 2K$. For example, in four dimensions, the Gauss-Bonnet combination will not have any contribution to the field equations and this fact leads to the Bach-Lanczos identity [Lanczos, 1938]

$$
C_{\alpha\mu\nu\lambda}C_{\beta}{}^{\mu\nu\lambda} = \frac{1}{4}g_{\alpha\beta}C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma}, \tag{I.19}
$$

where $C_{\mu\nu\lambda\sigma}$ is the Weyl tensor.

Now, let us consider an infinitesimal variation of the connection and vielbein forms. The corresponding variation of L_K is $\delta_e L_K + \delta_\omega L_K$.

For $1 \leq K \leq D/2$, variation of $\omega^A{}_B$ yields (using Eq.I.12)

$$
\delta_\omega L_K = K(\mathcal{D}\delta\omega^{A_1B_1})R^{A_2B_2}\dots R^{A_KB_K} \in_{A_1B_1\cdots A_KB_K}. \tag{I.20}
$$

Now, using Eq.(I.11),

$$d(\delta\omega^{A_1B_1}R^{A_2B_2}\dots R^{A_KB_K}\in_{A_1B_1\cdots A_KB_K})$$
$$=(\mathcal{D}\delta\omega^{A_1B_1})R^{A_2B_2}\in_{A_1B_1\cdots A_KB_K}+\delta\omega^{A_1B_1}R^{A_2B_2}\dots R^{A_KB_K}\mathcal{D}\in_{A_1B_1\cdots A_KB_K} \quad (I.21)$$

$$\text{So,}\qquad \delta_\omega L_K = K\delta\omega^{A_1B_1}R^{A_2B_2}\dots R^{A_KB_K}(\mathcal{D}\in_{A_1B_1\cdots A_KB_K})+\text{exact form} \qquad (I.22)$$

If $D=2K$, the above variation is in exact form. For $D>2K$,

$$\delta_\omega L_K = K\delta\omega^{A_1B_1}R^{A_2B_2}\dots R^{A_KB_K}T^C\in_{A_1B_1\cdots A_KB_KC}+\text{exact form} \qquad (I.23)$$

If we restrict our considerations to the pure metric theory where $T^C=0$, we see that that the variation of the connection does not contribute to the field equations : (if $D>2K$)

$$\sum_{K=0}^{p}\lambda_K\delta_e L_K = 0 \qquad (I.24a)$$

$$\text{where}\qquad \delta_e L_K = \delta e^C R^{A_1B_1}\dots R^{A_KB_K}\in_{A_1B_1\cdots A_KB_KC} \qquad (I.24b)$$

In particular, one gets

$$\begin{aligned}
\delta_e L_0 &= \delta e^A \in_A\\
\delta_e L_1 &= \delta e^A(Rg_{AB}-2R_{BA})\in^B\\
\delta_e L_2 &= \delta e^E\big[(R_{ABCD}R^{CDAB}-4R_{AB}R^{BA}+R^2)g_{EF}\\
&\qquad +4(R_{FCAB}R^{ABC}{}_E+2R_{FAEB}R^{BA}+2R_{FC}R^C{}_E-RR_{FE})\big]\in^F \qquad (I.25)\\
&\qquad \dots\text{etc.}
\end{aligned}$$

All these terms contribute higher order corrections to the field equations of these theories.

I.(D) Solutions of Higher Dimensional and Higher Order Gravity

Many known objects which naturally arise in ordinary GR have been studied in the context of higher dimensional theories with or without higher order terms. In the former case, the interest has been mainly centered on the simplest version of higher order gravity, i.e. the Einstein-Gauss-Bonnet (EGB) theory in which the Lagrangian is a sum of the Einstein-Hilbert and the Gauss-Bonnet terms. The action can thus be written as

$$I = \int d^D x \sqrt{-g}[R/\kappa + \alpha(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)], \qquad (I.26)$$

$$\text{where} \qquad \kappa = 4G \int d\Omega_n, \qquad n = D - 2.$$

$\int d\Omega_n$ is the area of a unit n-sphere and α is the string slope parameter with magnitude of the order of the square of the Planck length, and is positive as long as we consider EGB theory to be the low frequency limit of Superstring theory. However, if the EGB theory is assumed as a theory in its own right (i.e. Lovelock gravity), there is no restriction on the magnitude or the sign of α and it is viewed as just a coupling coefficient of higher order terms. Both the Newton's constant G and κ have dimensions $L^{(D-3)}/M$.

The field equations that follow from such an action are given as

$$\begin{aligned}
0 =& G_{\mu\nu}/\kappa - \alpha S_{\mu\nu} \\
=& \frac{1}{\kappa}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] - \alpha\Big[\frac{1}{2}g_{\mu\nu}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2) \\
& - 2RR_{\mu\nu} + 4R_{\mu\alpha}R^{\alpha}{}_{\nu} + 4R_{\alpha\beta}R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} - 2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma}\Big]. \qquad (I.27)
\end{aligned}$$

An action like (I.26) is of independent interest since this allows spontaneous compactification [Müller-Hoissen, 1985; Mignemi, 1986]. The multidimensional solutions which have been studied so far may be categorised into two classes:

(a) WITHOUT COMPACTIFIED DIMENSIONS :

In these cases, all dimensions are considered to be on equal footing. The generalised Einstein equations have as 'ground states' the maximally symmetric solutions– Minkowski

space, de Sitter space and Anti de Sitter space. So, the solutions asymptotically approach either the Minkowski space (M^4) if the spacetime becomes flat at infinity, or D-dimensional de Sitter or Anti de Sitter space if the spacetime remains curved at infinity. Although these solutions are irrelevent to present day low energy physics, but the study of such solutions may give us important insight into the concept of the existence of higher dimensions and may answer some questions related to the 'specificity' of the number of dimensions discussed in section I.(A).

Here we are mentioning various works done in this line :

(i) *Black Holes* : In ordinary (without higher order terms) higher dimensional gravity, both static and stationary solutions (charged or uncharged) have been studied [see Tangherlini,1963, Myers and Perry,1986, Dianyan,1988]. In higher order gravity, so far, only static spherically symmetric black holes seem to have been investigated [see Boulware and Deser,1985, Wheeler,1986, Myers,1987, Myers and Simon, 1988, Wiltshire,1988, Callan, Myers and Perry,1989]. Wiltshire (1986) did the electromagnetic extension of such solutions. The assumed topology for such solutions is $R^2 \times S^{D-2}$ and spherical symmetry in all the $(D-1)$ spatial dimensions.

(ii) *Gravitational Waves* and their propagation in higher order theory have been studied by Boulware and Deser (1985), Gibbons and Ruback (1986) and Tomimatsu and Ishihara (1987).

(iii) *Euclidean Wormholes* : Gonsalez-Diaz (1990) and Jianjun and Sicong (1991) studied Euclidean wormholes in the context of the EGB theory.

(iv) *Lorentzian Wormholes* : Bhawal and Kar (1992) first investigated the possibility of the existense of Lorentzian Wormhole solutions in EGB theory. They have shown that similar to the situation in four dimensional GR, the matter that threads the wormhole violates the Weak Energy Condition for positive values of α. For negative values of α, the condition may or may not be violated. They have also suggested the possible construction of a solution with matter satisfying Weak energy condition everywhere.

(b) WITH COMPACTIFIED EXTRA DIMENSIONS :

This class of solutions is based on the Kaluza-Klein view of the world geometry and more relevent to our present day low energy physics where the extra dimensions are unobservable. All cosmological solutions belong to this class. The main purpose for developing them is to find the reasons for a phase transition and the consequent dynamical reduction mechanism which may account for the huge discrepency of scales between the three observed dimensions and the extra microscopic ones. These models may also explore the possibility of solving horizon and flatness problems which arise in the standard cosmology. Thus, these also appear to be possible alternatives to the usual inflationary models. When the ordinary dimensions increase, the extra dimensions decrease and with them the mean volume. This corresponds to the increase in temperature which, in turn, may be interpreted to be an increase in the entropy of the universe.

Ordinary higher dimensional Kaluza-Klein cosmological models have been studied by several authors [see Sahdev, 1984 and references therein. Also see Appelquist et al.,1987 for a collection of papers]. In the context of higher order theories such models were studied by Shafi and Wetterich(1985), Madore(1985,1986), Yoshimura1986), Wheeler(1986), Ishihara(1986), Maeda(1986), Müller-Hoissen(1986), Henriques (1986)[but initial field equations(2.7) written in this paper are not correctly written], Deruelle and Madore(1987), Mukherjee and Paul(1990, Barrow and Cotsakis(1991) etc.

In the next chapter, we shall study and discuss different classical and semiclassical aspects of a simple model of this theory— a static, spherically symmetric solution— the Boulware Deser black hole.

Chapter II

THE BOULWARE-DESER BLACK HOLE

In this chapter, we concentrate our study on a static spherically symmetric black hole solution of the EGB theory given by Boulware and Deser (1985). This solution may be thought to be the extension of higher dimensional Schwarzschild black hole to higher order theory. Therefore, study of such a simple model may reveal the nature of physical processes involved in a higher order gravity model. We shall specifically study two cases, i.e. Geodesic motion and Hawking radiation, and try to see to what extent the extra dimensions and/or higher order terms affect these aspects.

The metric element of this exact solution of the field equations (I.27) is written as (in unit $c = 1$)

$$ds^2 = -P\, dt^2 + P^{-1}\, dr^2 + r^2\, d\Omega_n^2, \tag{II.1}$$

$$\text{where} \qquad P = 1 + \frac{r^2}{2\bar{\alpha}\kappa}\left[1 - \left[1 + \frac{8GM\bar{\alpha}\kappa}{r^{n+1}}\right]^{1/2}\right], \tag{II.2}$$

$$\text{and} \quad \bar{\alpha} = \alpha(n-1)(n-2), \qquad n = D - 2.$$

$d\Omega_n^2$ is the surface element of a unit n-sphere :

$$d\Omega_n^2 = d\theta_n^2 + \sin^2\theta_n[\,d\theta_{n-1}^2 + \sin^2\theta_{n-1}[\,d\theta_{n-2}^2 + \cdots + \sin^2\theta_3(\,d\theta_2^2 + \sin^2\theta_2\, d\theta_1^2))\cdots]] \tag{II.3}$$

$$\text{where} \quad 0 \le \theta_1 < 2\pi, \quad 0 \le \theta_i < \pi, \quad i = 2, 3, \ldots, n.$$

If we choose units $c = G = 1$, then $[M] \sim L^{n-1}$ and P can be written in terms of dimensionless variables

$$\rho = r(M)^{-1/(n-1)}, \tag{II.4a}$$

$$a = \bar{\alpha}\kappa(M)^{-2/(n-1)} \tag{II.4}$$

The variable ρ is our new dimensionless radial coordinate, so that

$$P = 1 + \frac{\rho^2}{2a}\left[1 - \left[1 + \frac{8a}{\rho^{n+1}}\right]^{1/2}\right]. \qquad (II.5)$$

In the limit $a \to 0$, P can be written in its limiting form

$$P = 1 - \frac{2}{\rho^{n+1}}. \qquad (II.6)$$

This represents the higher-dimensional Schwarzschild solution, which we would have obtained in its 'exact' form, if we had started with the Einstein action of ordinary D-dimensional spacetime :

$$I = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\,R. \qquad (II.7)$$

The horizon is given by $P = 0$ and, for a Boulware-Deser black hole, this is located at $\rho = \rho_h$ or $r = r_h$ and is given by

$$\begin{aligned} &\rho_h^{n-1} + a\rho_h^{n-3} - 2 = 0 \\ \text{or,} \quad &r_h^{n-1} + \bar{\alpha}\kappa r_h^{n-3} - 2GM = 0 \end{aligned} \qquad (II.8)$$

Correspondingly, for the higher-dimensional Schwarzschild solution (a=0), the horizon is located at

$$\rho_h = 2^{1/(n-1)} \quad \text{or,} \quad r_h = (2GM)^{1/(n-1)}. \qquad (II.9a)$$

For $a > 0$, there is only one horizon. For the five dimensional Boulware Deser solution the horizon is at

$$\rho_h = \sqrt{2-a} \qquad \text{or} \qquad r_h = \sqrt{2GM - \bar{\alpha}\kappa}, \qquad (II.9b)$$

so that we have a black hole solution only if $a < 2$.

II.(A) Geodesic motion

This section contains the results of our investigation on the geodesic motion in the Boulware- Deser Black Hole (BDBH) spacetime [Bhawal, 1990]. The study of geodesics is very useful for probing into the geometry of a spacetime. In the present context, this provides us information about the effect of the presence of extra dimensions and of the higher derivative terms on the motion of geodesics. We study only the cases corresponding to the positive values of α. Since this solution is asymptotic to the Schwarzschild spacetime of corresponding dimensions [Tangherlini,1963], we also compare the results of the two cases. This will help us in isolating the effects originating due to the presence of higher derivative terms from those present in ordinary higher dimensional gravity.

The symmetries of the Boulware-Deser spacetime give us ${}^{n+1}C_2$ 'rotational' Killing vectors and one 'time-translational' Killing vector given respectively as

$$\xi_{ij} = x_{[i}\partial_{j]}, \qquad i,j = 1,2,\ldots,(n+1) \tag{II.10}$$

$$\xi_t = \partial_t. \tag{II.11}$$

Note that different combinations of the assigned values of the indices i,j correspond to different Killing vectors and not their tensor components.

The solution of the geodesic equations is then considerably facilitated if we employ integrals of motion by using the theorem that if $\mathbf{u}$ is the $(n+2)$-velocity of a geodesic, then for any Killing vector ξ, we have a constant of motion $\xi.\mathbf{u}$.

In terms of the polar coordinates, X^μ, we have $u_\mu = g_{\mu\nu}(dX^\mu/d\tau)$ where τ is the proper time along the geodesic. So, from the Killing vector ξ_{t}, we obtain the constant E :

$$\xi_{\mathrm{t}}.\mathbf{u} = u_t = -P\frac{dt}{d\tau} = -E \tag{II.12}$$

Similarly, the rotational Killing vectors give us constants ℓ_{ij},

$$\xi_{\mathrm{ij}}.\mathbf{u} = \ell_{ij} \tag{II.13}$$

These are analogous to angular momenta. This leads us to define the 'total angular momentum' to be

$$L^2 = \sum_{i<j} \ell_{ij}^2 = \rho^2[u^2 - (\mathrm{u}.\hat{e}_\rho)^2 + (\mathrm{u}.\hat{e}_t)^2] \qquad (II.14)$$

$\hat{e}_\rho$, $\hat{e}_t$ being unit vectors in ρ and t directions respectively. So,

$$L^2 = \rho^2\left[u^2 - \frac{\dot{\rho}^2}{P} + \frac{E^2}{P}\right], \qquad (II.15)$$

where 'dot' denotes differentiation with respect to τ. The quantity E is defined to be the energy. Rearranging, we get

$$-\frac{E^2}{P} + \frac{\dot{\rho}^2}{P} + \frac{L^2}{\rho^2} = u^2 = \begin{cases} -1 & \text{for 'timelike' geodesics} \\ 0 & \text{for 'null' geodesics} \end{cases} \qquad (II.16)$$

Since this solution is spherically symmetric, we can always describe the geodesics in an invariant hyperplane described by all $\theta_k = \pi/2$, $k = 2, 3, \ldots n$. Then the Killing vector corresponding to the symmetry in the direction of the azimuthal angle θ_1 gives us

$$\rho^2\dot{\theta}_1 = L \qquad (II.17)$$

The set of three equations II.12, 16 & 17 describes the motion of geodesics in Boulware-Deser black hole spacetime.

(i)*Timelike Geodesics* ($u^2 = -1$)

Considering the above equations of motion, the behaviour of the trajectory of a particle both in proper time and coordinate time can be investigated. The situation in the case of an infalling particle has been illustrated in Figure 1 for a particular combination of values of 'E' and 'a' [$E = 2$, $a = 1$, ρ(initial) $= 10$]. As usual, with respect to an observer stationed at infinity, the particle describing a timelike trajectory will take an infinite time to reach the horizon, even though it will cross the horizon and arrive at the singularity in a finite proper time. It is found that as 'a' decreases, the finite proper time needed to reach the singularity also reduces. This reduction is however very small as compared to the usual value of proper time needed.

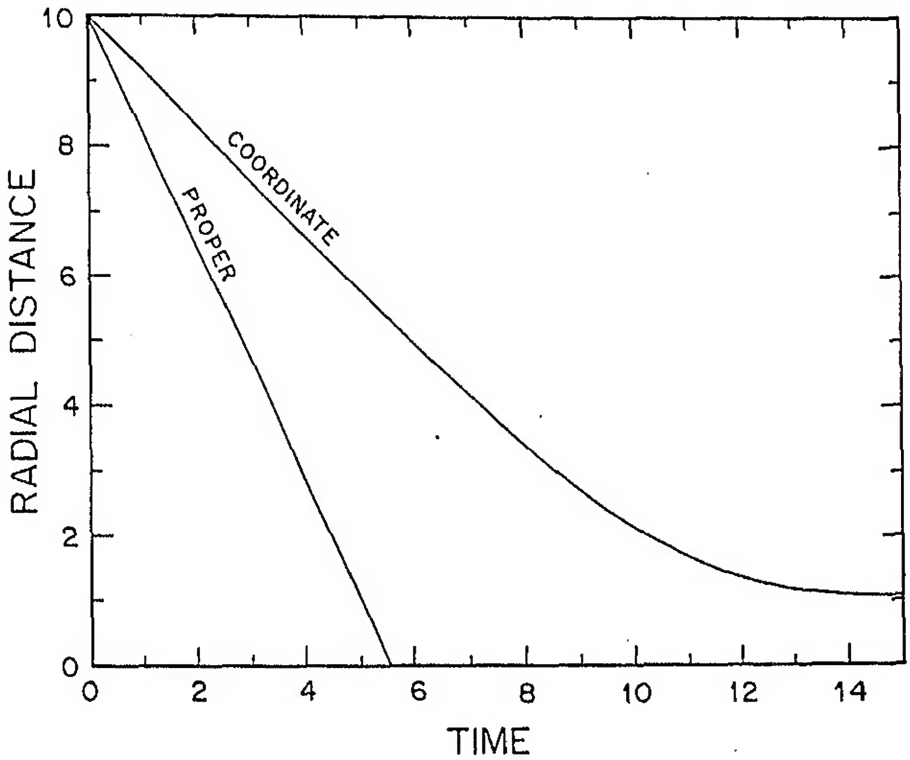

Fig.1 *Time in 5-dimensional BDBH.*

The field equation (II.16) may be written as

$$\frac{1}{2}\dot{\rho}^2 + \frac{1}{2}(1 + \frac{L^2}{\rho^2})P = \frac{1}{2}E^2. \qquad (II.18)$$

This shows that the radial motion of a geodesic is the same as that of a unit mass particle of energy $E^2/2$ in ordinary one dimensional non-relativistic mechanics moving in the effective potential

$$V_{\text{eff}} = \frac{1}{2}\left[1 + \frac{L^2}{\rho^2}\right]P. \qquad (II.19)$$

In the case of the five-dimensional Schwarzschild black hole,

$$V_{\text{eff}} = \frac{1}{2} - \frac{1}{R^2} + \frac{L^2}{2R^2} - \frac{L^2}{R^4}, \qquad (II.20)$$

the extremum of which is at $\rho_e = 2L/(L^2 - 2)^{1/2}$.

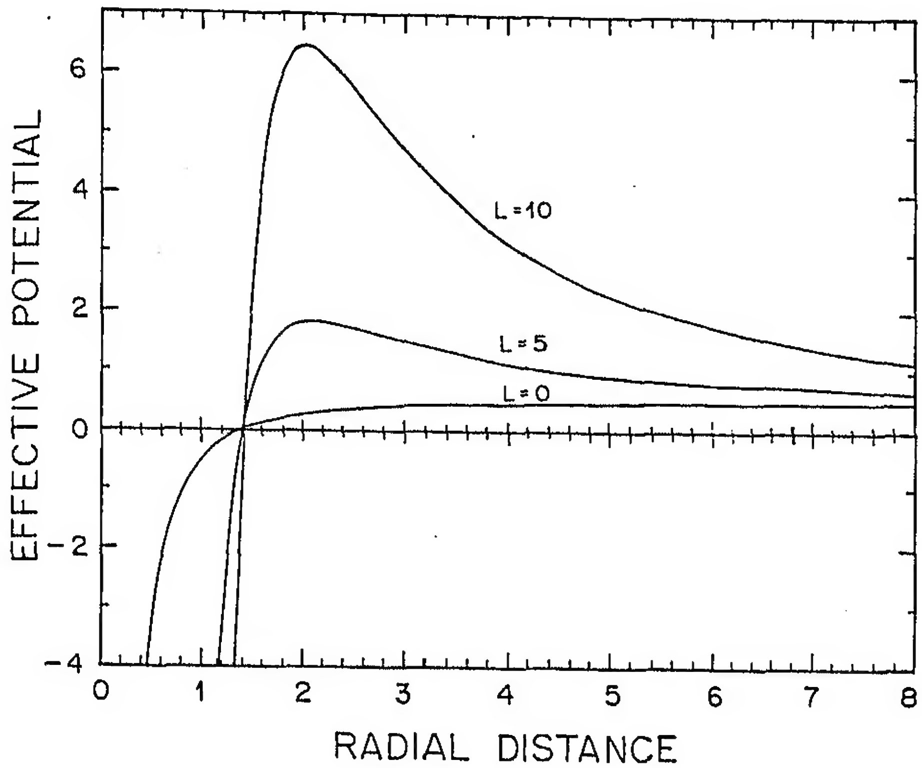

Fig.2 V_{eff} *for timelike geodesics in 5-dim. Schwarzschild black hole* $(a = 0)$

If $L^2 < 2$, there is no extremum of V_{eff}. A particle heading toward the center of attraction with $L^2 < 2$ will fall directly to the horizon and will continue its fall into the spacetime singularity at $\rho = 0$. From now on, we shall refer to such a monotonous type of potential as type I.

If $L^2 > 2$, we can easily see that the extremum ρ_e is the maximum point of V_{eff}. After that, the potential dies down to 0.500 as $\rho \to \infty$. From now on, we shall refer to such type of potential as type II.

Figure 2 shows different V_{eff} vs ρ plots obtained for different values of L. The lowest one ($L = 0$) is of type I. No stable bound orbit is possible. For a type II potential, only an unstable circular orbit can exist at ρ_e.

In a five dimensional Boulware-Deser black hole,

$$V_{\text{eff}} = \frac{1}{2} + \frac{L^2}{4a} + \frac{\rho^2}{4a} + \frac{L^2}{2\rho^2} - \frac{\rho^2 + L^2}{4a}\left[1 + \frac{8a}{\rho^4}\right]^{1/2} \tag{II.21}$$

The extremum point ($\rho = \rho_e$) of this potential is given by the equation

$$\rho_e^8(2 - L^2) + 4L^2\rho_e^6 + \rho_e^4(aL^4 - 8aL^2) + 8aL^4(a - 2) = 0 \tag{II.22}$$

The solution of this quartic equation in ρ_e^2 is very complicated and can be written as

$$\begin{aligned}
\rho_e =& \sqrt{x} \\
x =& \frac{-(p-l) \pm [(p-l)^2 - 4(k-b)]^{1/2}}{2} \quad \text{and} \quad \frac{-(p+l) \pm [(p+l)^2 - 4(k+b)]^{1/2}}{2},
\end{aligned}$$

where

$$\begin{aligned}
l =& (p^2 + 2k - q)^{1/2}, \qquad b = (k^2 - s)^{1/2} \\
k =& \left[-\frac{\beta}{2} + \left[\frac{\beta^2}{4} + \frac{\sigma^3}{27}\right]^{1/2}\right]^{1/3} + \left[-\frac{\beta}{2} - \left[\frac{\beta^2}{4} + \frac{\sigma^3}{27}\right]^{1/2}\right]^{1/3} + \frac{q}{6}, \\
\sigma =& -\left[\frac{q^2}{12} + s\right], \qquad \beta = \frac{1}{3}sq - \frac{5}{432}q^3 - \frac{1}{2}p^2 s, \\
p =& \frac{2L^2}{2 - L^2}, \qquad q = \frac{aL^4 - 8aL^2}{2 - L^2}, \qquad s = \frac{8aL^4(a-2)}{2 - L^2}.
\end{aligned} \tag{II.23}$$

From this solution, it is very difficult to make a general statement on the shape of the V_{eff} vs ρ characteristics in different cases. However, we have plotted these curves for different values of a and L, a typical one of which is shown in Figure 3. In all cases, the effective potential curves are either of type I or II. Therefore, in no case, a stable bound orbit is possible. For a type II potential an unstable circular orbit can exist at a radius $\rho = \rho_e$. The five velocity of this circular orbit can be represented as

$$u^\alpha = N(1, 0, 0, 0, \omega), \tag{II.24}$$

where

$$\omega^2 = \frac{1}{2a}\left[1 + \left[1 + \frac{8a}{\rho_e^4}\right]^{-1/2}\right] \quad \text{and} \quad N^{-2} = 1 - \frac{4}{\rho_e^2}\left[1 + \frac{8a}{\rho_e^4}\right]^{-1/2}.$$

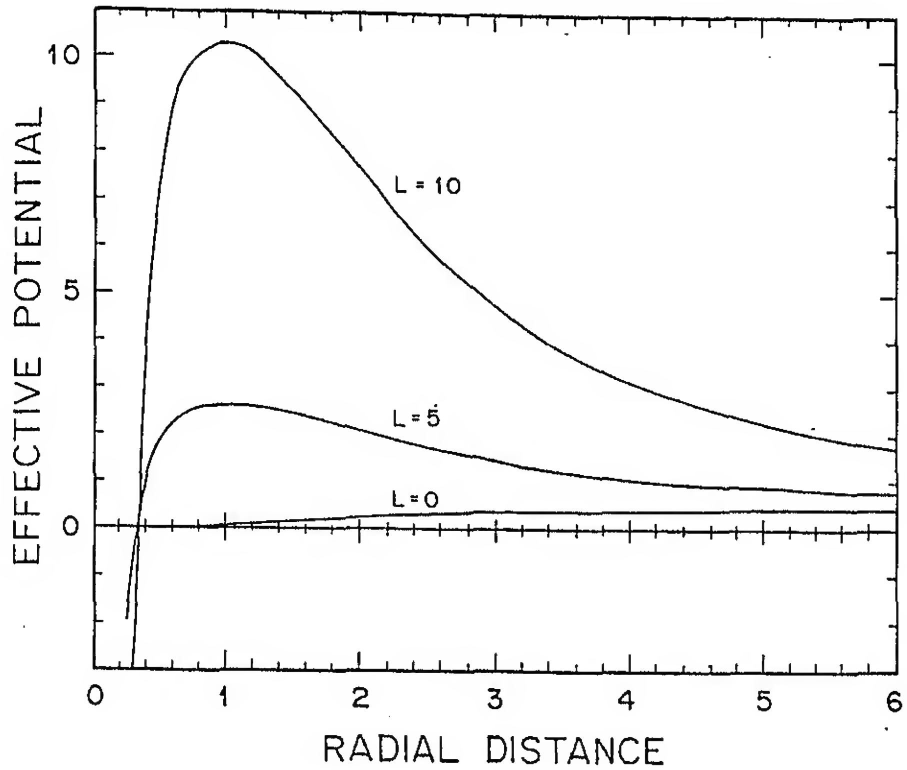

Fig.3 *V_{eff} for timelike geodesics for a constant value of $a(=1.9)$ in 5-dim. BDBH*

For a fixed a as L increases, the height of the maximum increases. The effective potential becomes zero at the horizon $\rho_h = \sqrt{2-a}$ and, in all cases, asymptotically tends to 0.500 as ρ increases. For a fixed L, as a increases, the horizon consequently shifts to the left as does the location of the maximum (Figure 4).

We have also investigated the field equations in other higher dimensions ($D > 5$). But the results obtained are of similar nature as those in five dimensions. Again, the effective potential is either of type I or II.

(ii)*Null Geodesics* (u=0)

For a five dimensional Schwarzschild black hole,

$$V_{\text{eff}} = \frac{L^2}{2\rho^2} - \frac{L^2}{\rho^4}. \qquad (II.25)$$

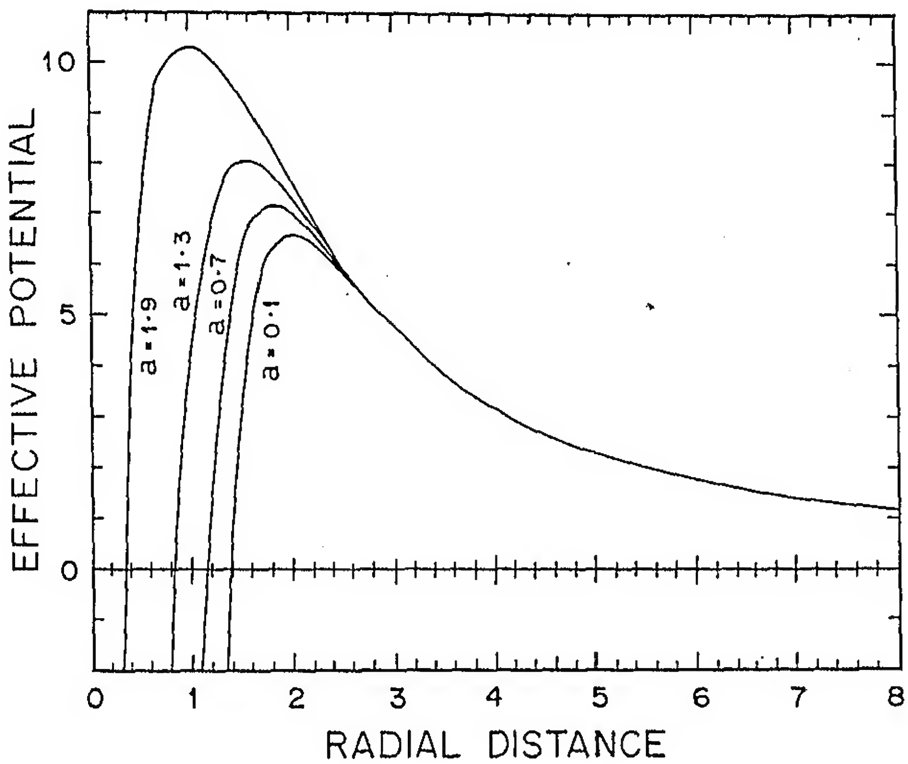

Fig.4 *V_{eff} for timelike geodesics for a constant value of $L(=10)$ in 5-dim. BDBH*

In this case, unlike the timelike geodesics, the potential is always of type II and the maximum point is at $\rho_e = 2$. The maximum value of the effective potential is dependent on L from which we deduce that the five dimensional Schwarzschild geometry will capture any photon sent toward it with an apparent impact parameter smaller than the critical value $b_c = 2\sqrt{2}$. The effective potential for null geodesics in five dimensional Boulware-Deser geometry is

$$V_{\text{eff}} = \frac{L^2}{2\rho^2} + \frac{L^2}{4a} - \frac{L^2}{4a}\left[1 + \frac{8a}{\rho^4}\right]^{1/2}. \qquad (II.26)$$

The position of the maximum is at $\rho_e = (16 - 8a)^{1/4}$. So, in this case also, for any value of L or a $(0 < a < 2)$, the potential is of type II (Figures 5 & 6). The critical value of the 'apparent impact parameter' can be calculated to be

$$b_c = \left[\frac{2a\sqrt{2}}{\sqrt{2} - \sqrt{2 - a}}\right]^{1/2}. \qquad (II.27)$$

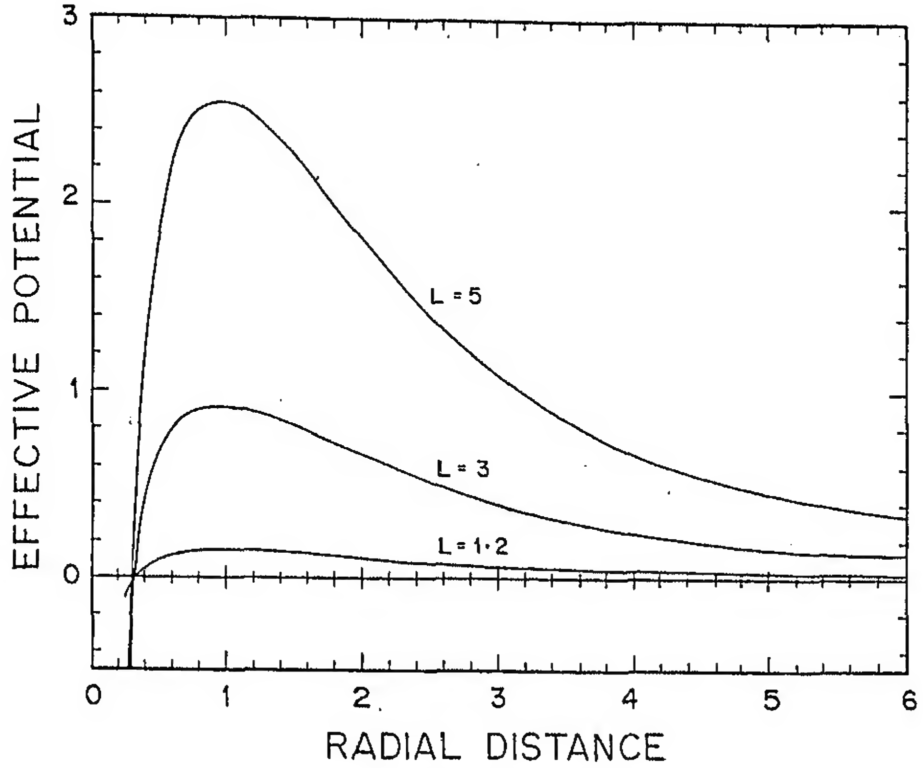

Fig.5 *V_{eff} for null geodesics for a constant value of $a(=1.9)$ in 5-dim. BDBH*

(iii) *Conclusions*

We have observed that the nature of the effective potential curves for five dimensional Boulware-Deser spacetime is not different from those in five dimensional Schwarzschild solution. If we compare the two cases, we see that for a low or high value of L, the presence of a nonzero a (and, consequently, the presence of higher order terms in the action of the field) does not significantly affect the nature of geodesic orbits, except changing the position of the horizon and the maximum point of effective potential. However, for values of L around $L = \sqrt{2}$, the presence of a (or, sometimes, a high value of a) may change a type I potential to a type II one, being determined by Eq.(II.23).

It is a well-known result that in four dimensional Schwarzschild black hole geometry, stable bound orbits are possible for timelike geodesics. On the other hand, our numerical

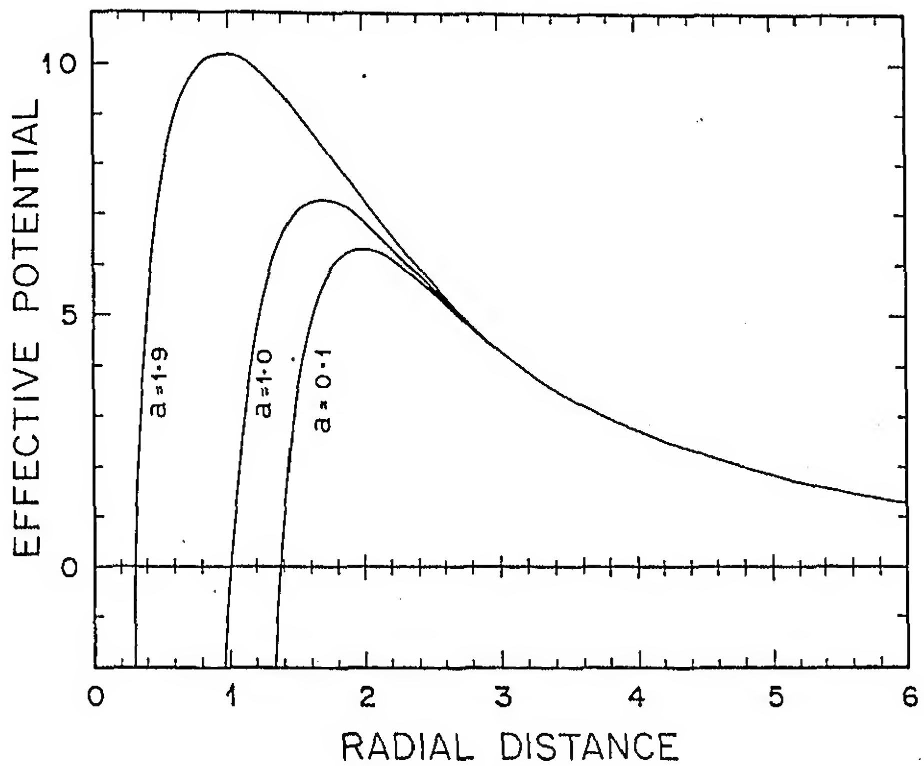

Fig.6 *V_{eff} for null geodesics for a constant value of $L(=10)$ in 5-dimensional BDBH*

solutions have shown that the same is not true for higher dimensional Schwarzschild or Boulware-Deser spacetimes, where only unstable orbits are possible for suitable values of L. Considering the analytical solutions obtained, it is not very easy to arrive at the above conclusion because of the complexity of these solutions. However, drawing analogy with Newtonian mechanics, we can expect the above results.

It is well known that , also in Newtonian mechanics, a particle in a potential of the form $\phi \propto 1/r^{n-1}$ $(n > 2)$ cannot describe a stable bound orbit*. The underlying connection between this fact and the present results can be readily understood. In the weak field limit, $g_{00} = -(1 + 2\phi)$. In the higher dimensional Schwarzschild solution, by Newtonian approximation, ϕ is taken to be $-GM/r^{(n-1)}$. The absence of stable bound orbits in higher

* A corollary of the Bertrand's theorem

dimensional Scharzschild or Boulware-Deser spacetimes is the general relativistic analog of the above Newtonian result. This also indicate that $D = 5$ behaviour is generic because by this argument, the behaviour of geodesics for any $n \geq 3$ will always be the same.

(B) Hawking Temperature

The discovery by Hawking (1974,1975) that black holes can radiate was a startling one. In this section, we apply similar techniques to obtain an expression for the Hawking temperature of Boulware-Deser black hole. This will reveal the effect of the presence of extra dimensions and higher order terms on these quantum mechanical calculations. As we shall see, the techniques used in the four-dimensional case can be extended to any higher dimensional spherically symmetric black hole solution in a straight forward manner.

The calculations related to Hawking radiation are based on the nonperturbative techniques, i.e. Bogolubov transformation, in the context of the semiclassical approach to gravitation or 'Quantum Field Theory in Curved Spacetime'. The technicalities involved are standard ones and, by now, quite familiar in the literature [Birrel and Davies, 1982]. That is why we do not formally discuss those methods here. However, we attempt to present the topic in a self-contained manner. Throughout this section we work with the units in which $\hbar = c = k_B = 1$.

The starting point is the scalar mode solutions of the Klein-Gordon Equation

$$\frac{1}{\sqrt{-g}}[\sqrt{-g}g^{\mu\nu}\Phi_{,\nu}]_{,\mu} = 0. \tag{II.28}$$

These solutions were first given by Iyer,Iyer and Vishveshwara(1989) in the form :

$$\Phi = \frac{1}{r^{n/2}}\mathcal{R}(r)e^{-i\omega t}\mathcal{A}(\theta_n, \ldots, \theta_1), \tag{II.29}$$

$$\text{where} \qquad \mathcal{A} \equiv e^{i\ell_1\theta_1}\prod_{i=2}^{n}(1 - z_i^2)^{\ell_{i-1}/2}C_{\lambda_i}^{\mu_i}(z_i) \tag{II.30}$$

$$\text{and} \quad z_i = \cos\theta_i$$

$$\mu_i = \frac{1}{2}\ell_{i-1} + \frac{1}{2}(i-1)$$

$$\lambda_i = \ell_i - \ell_{i-1}$$

Also, ℓ_1 and ℓ_i are integers. All $\ell_i \geq 0$ and

$$| \ell_1 | \leq \ell_2 \leq \ldots \leq \ell_n.$$

$C^{\mu_i}_{\lambda_i}(z_i)$ are Gegenbauer functions.

The radial function $\mathcal{R}(r)$ satisfies

$$\frac{d^2\mathcal{R}}{dr^{*2}} + (\omega^2 - V(r))\mathcal{R} = 0 \tag{II.31}$$

where r^* is defined to be

$$\frac{dr^*}{dr} = \frac{1}{P} \tag{II.32}$$

$$V(r) = e^{-2\lambda}\left[\frac{\bar{\ell}_n^2}{r^2} + \frac{n(n-2)}{4r^2}P + \frac{n}{2}\frac{P_{,r}}{r}\right]. \tag{II.33}$$

In the asymptotic region ($r \to \infty$ or $r^* \to \infty$), Eq.(II.29) reduces to

$$\frac{N}{\omega^{1/2}r^{1/2}}\, e^{-i\omega u}\, \mathcal{A}(\theta_1, \theta_i) \tag{II.34}$$

$$\text{and} \qquad \frac{N}{\omega^{1/2}r^{1/2}}\, e^{-i\omega v}\, \mathcal{A}(\theta_1, \theta_i) \tag{II.35}$$

in terms of the null coordinates $u = t - r^*$ and $v = t + r^*$. N is the normalization constant.

At early times ($t \to -\infty$), the solutions $f_{\omega\ell}$ of the wave equation can be chosen so that on past null infinity $\mathcal{I}^-$ they form a complete family satisfying orthonormality conditions

$$(f_{\omega\ell}, f_{\omega'\ell'}) = \delta(\omega - \omega')\delta_{\ell_1,\ell_1'} \ldots \delta_{\ell_n \ell_n'}. \tag{II.36}$$

In our compact notation the index $\ell \equiv (\ell_1, \ell_2, \ldots, \ell_n)$.

They contain only positive frequency modes and are chosen to reduce to the incoming spherical modes (II.35) in the remote past. The field Φ can be decomposed as

$$\Phi(-\infty) = \sum_{\ell} \int (a_{\omega\ell} f_{\omega\ell} + a^{\dagger}_{\omega\ell} f^*_{\omega\ell}) d\omega. \tag{II.37}$$

The 'in vacuum' state corresponding to the absence of incoming radiation from $\mathcal{I}^-$ can be defined as

$$a_{\omega\ell} \mid 0 >_{\text{in}} = 0 \qquad \text{for all } \omega,\ \ell. \tag{II.38}$$

Because of the presence of the potential $V(r)$ in Eq.(II.31), the standard incoming waves (II.35) will be partially scattered back by the background field to become a superposition of incoming and outgoing waves. These outgoing modes are totally different from those which arise as a result of the passage of incoming modes through the interior of the collapsing ball to the opposite side.

Since the interesting thermal effects arise only from the latter contribution, for the time being, we will remove $V(r)$ from Eq.(II.31), so that the field modes can be simply given by Eqs.II.34 and 35 everywhere.

At late times, the field is described by the superposition of two types of modes. First, there are the outgoing modes (II.34) which we call $p_{\omega\ell}$. At future null infinity $\mathcal{I}^+$, these are the positive frequency modes. Also, there will always be modes incoming at the event horizon which we call $q_{\omega\ell}$. So, at late times, the field is expanded as

$$\Phi(-\infty) = \sum_{\ell} \int d\omega \{b_{\omega\ell} p_{\omega\ell} + \text{h.c} + c_{\omega\ell} q_{\omega\ell} + \text{h.c.}\} \tag{II.39}$$

h.c→ hermitian conjugate.

Since massless fields are completely determined by their data on the past null infinity $\mathcal{I}^-$, one can express both $p_{\omega\ell}$ and $q_{\omega\ell}$ as linear combinations of $f_{\omega\ell}$ and $f^*_{\omega\ell}$. Let

$$p_{\omega\ell} = \sum_{\ell} \int d\omega' (\alpha_{\omega\omega'} f_{\omega'\ell} + \beta_{\omega\omega'} f^*_{\omega'\ell}). \tag{II.40}$$

This is known as Bogolubov transformation. The coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are known as Bogolubov coefficients. These satisfy the normalization condition (or the Wronskian condition) :

$$1 = \int d\omega' (\mid \alpha_{\omega\omega'} \mid^2 - \mid \beta_{\omega\omega'} \mid^2). \tag{II.41}$$

Therefore, the 'in vacuum' state will not appear to be a vacuum state to an observer at $\mathcal{I}^+$. Instead he will find that the expectation value of the Number operator in the 'in vacuum' to be

$$N_\omega = {}_{\text{in}}< 0 \mid b^\dagger_{\omega\ell} b_{\omega\ell} \mid 0 >_{\text{in}} = \int d\omega' \mid \beta_{\omega\omega'} \mid^2 \tag{II.42}$$

$$\text{where} \qquad \beta_{\omega\omega'} = (p_{\omega\ell}, f^*_{\omega'\ell}). \tag{II.43}$$

Thus, in order to determine the number of particles created by the gravitational field and emitted to infinity, one has to calculate $\beta_{\omega\omega'}$.

To evaluate $\beta_{\omega\omega'}$, it was found to be convenient to take the surface of integration to lie in the in-region. This corresponds to the modes $p_{\omega\ell}$ being traced back along the null path. The modes $p_{\omega\ell}$ is of the form (II.34). At early times, the ray will be moving along constant v lines. But since the phase of the wave remains constant, it will still have the numerical value $e^{-i\omega u(v)}$ where the function $u(v)$ has to be determined. Following Hawking (1974), one may show that

$$u = -\ln[v_0 - v]/S + \text{constant} \tag{II.44}$$

where v_0 is the value of the ray surface that forms the event horizon. The quantity S is called 'Surface Gravity' and is given as

$$S = \frac{1}{2}\frac{dP}{dr}\bigg|_{r=r_h}. \tag{II.45}$$

r_h describes the horizon of the black hole where $P = 0$. Another derivation of Eq.(II.44) using moving mirror consideration can be found in Birrel and Davies (1982).

So, at early times, we have

$$p_{\omega\ell} = \begin{cases} N\omega^{-1/2}r^{-1}\mathcal{A}\exp\left[\frac{i\omega}{S}\ln(v_0 - v)/K\right], & \text{for } v < v_0 \\ 0 & \text{for } v > v_0. \end{cases} \tag{II.46}$$

where K is a constant.

Now, the ordinary in-vacuum is defined with respect to modes $f_{\omega\ell}$ given by Eq.(II.35). So, using Eq.(II.43), the Bogolubov coefficients can be determined as

$$\beta_{\omega\omega'} = \frac{1}{2\pi}\int_{-\infty}^{v_0} dv\left(\frac{\omega'}{\omega}\right)^{1/2} e^{-i\omega' v}\exp\left[\frac{i\omega}{S}\ln(v_0 - v)\right]. \tag{II.47}$$

One may also calculate

$$\alpha_{\omega\omega'} = (p_{\omega\ell}, f_{\omega'\ell})\frac{1}{2\pi}\int_{-\infty}^{v_0} dv\left(\frac{\omega'}{\omega}\right)^{1/2} e^{+i\omega' v}\exp\left[\frac{i\omega}{S}\ln(v_0 - v)\right]. \tag{II.48}$$

These integrals can be evaluated in terms of Γ-functions. Using Eq.(II.41) one can have

$$| \alpha_{\omega\omega'} |^2 = \exp\left(\frac{2\pi\omega}{S}\right) | \beta_{\omega\omega'} |^2 \tag{II.49}$$

Now, using Wronskian condition(II.41), we can obtain from Eq.(II.42)

$$N_\omega = \left[\exp\left(\frac{2\pi\omega}{S}\right) - 1\right]^{-1}. \tag{II.50}$$

This corresponds to a Planck spectrum with temperature given by

$$T = \frac{S}{2\pi}. \tag{II.50}$$

Therefore, we observe that the most important quantity in this treatment is the surface gravity S and the Hawking temperature of any black hole of topology $R^2 \times S^{D-2}$ of the form(II.1) can be given by Eq.(II.50).

In five dimensional Boulware-Deser black hole given by Eq.(II.2),

$$S = \frac{\sqrt{2GM - \bar{\alpha}\kappa}}{(2GM + \bar{\alpha}\kappa)} \tag{II.51}$$

In general, the temperature of a D-dimensional Boulware-Deser black hole is

$$T = \frac{(D-5)GM + r_h^{D-3}}{2\pi r_h(4GM - r_h^{D-3})} \tag{II.52}$$

where r_h is a solution of Eq.(II.8).

Also, the temperature of an ordinary higher dimensional Schwarzschild black hole can be obtained by setting $\alpha = 0$

$$T = \frac{D-3}{4\pi(2GM)^{1/(D-3)}}. \tag{II.53}$$

Similar expressions were also obtained by Myers and Simon (1988) and Wiltshire (1988) by making use of an alternative method which is described below in a sketchy way. [There are ,however, some minor differences in the equation for r_h and the expression for T

because the definition of the constants and parameters of the black hole solution considered by them differ from those in the actual solution given by Boulware and Deser (1985).] By now, this method is also quite familiar in the literature [see Birrel and Davies,1982; Narlikar and Padmanabhan,1986].

The Hawking temperature of the spacetime metrics of the form (II.1) may be determined in each case by noting that if one analytically continues the metric to imaginary time, $t \rightarrow i\tau$, then the resulting manifold is regular if τ is identified with a period $\beta = 2\pi/S$.

After the analytical continuation, the metric (II.1) can be written with a Kruskal-like line element as

$$ds^2 = Pe^{-2Sr^*}[dX^2 + dY^2] + r^2 d\Omega_n^2 \qquad (II.54)$$

$$\text{where} \qquad X = S^{-1} e^{Sr^*} \cos S\tau$$
$$Y = S^{-1} e^{Sr^*} \sin S\tau \qquad (II.54)$$

This is a positive definite Riemannian space with topology $\mathcal{R}^2 \times \mathcal{S}^2$. Since the event horizon ($r = 2M$) is represented by the origin of the coordinate system, the singularity and the space inside the black hole are absent here. There is a rotational symmetry corresponding to the Killing vector ∂_τ. This endows τ with the properties of an angular coordinate with a periodicity $2\pi/S$. This periodicity in τ attributes the analytically continued Hartle-Hawking propagator [Hartle and Hawking,1976] all the properties of a thermal Green's function [Gibbons and Perry,1976,1978] from which the temperature can be easily identified to be $2\pi/S$.

Now, we may look back and correct the neglect of back scattering that depletes the outgoing flux by a factor $\mathcal{R}_\omega$, the reflection coefficient. This has the effect of replacing the left hand side of Eq.(II.41) by $(1 - \mathcal{R}_\omega)$. If we consider a sphere of radius r_0 centered on the collapsing ball, the density of states inside it will be $r_0 d\omega/(2\pi)$ [for all fixed ℓ_1, ℓ_i]. So, the number of particles per unit time in the frequency range ω to $\omega + d\omega$ passing out through the surface of the sphere can be calculated to be

$$\tilde{N}_\omega = \frac{d\omega}{2\pi} \frac{(1 - \mathcal{R}_\omega)}{(\exp(2\pi\omega/S) - 1)}. \qquad (II.55)$$

Since $\mathcal{R}_\omega$ is a function of ω, the spectrum is not precisely Planckian in nature. The total luminosity of the black hole can be found by integrating (II.55) over all modes. The numerical study of the dependence of $\mathcal{R}_\omega$ on ω for five and six dimensional Boulware-Deser black hole can be found in Iyer, Iyer and Vishveshwara(1988).

At the end of this section, we indicate here an interesting property of the five dimensional BDBH which one can guess from the expression of temperature in Eq.(II.51).

A problem faced in the process of Hawking radiation is that a black hole may vanish after radiating away its entire mass. In that case the incoming pure state fully converts to a mixed thermal state from the point of view of an observer located outside the event horizon, thus violating basic principle of quantum coherence— the time evolution should be described by a unitary operator, the Hamiltonian.

As is evident from Eq.(II.51), however, in this case, the black hole temperature may vanish at a finite mass and the evolution may end up with a zero temperature soliton. But the horizon also vanishes in that limit ($2GM \to \bar{\alpha}\kappa$) revealing the naked singularity. In this instance, then, the cosmic censorship hypothesis [Penrose, 1979] can be identified with the third law of black hole thermodynamics [Israel, 1986] which ensures that "No continuous process in which the energy tensor of accreted matter remains bounded and satisfies the weak energy condition in a neighbourhood of the apparent horizon can reduce the surface gravity of a black hole to zero within a finite advanced time". Such a black hole may, therefore, continue to radiate for ever without being fully evaporated away.

The Eq.(II.52) shows that a zero-temperature situation will never arise in BDBH solutions for $D > 5$. As shown by Myers and Simon (1988), this feature is always encountered in $2K + 1$ dimensions for a Lovelock theory including $2K$ dimensional Euler density.

II.(C) Comments on Back Reaction

It is a natural question to ask whether the semiclassical effects or the leading order corrections can solve the problem of singularity faced in ordinary GR. Classically speaking, in higher dimensions, at short distances, Gravity is more attractive than in four dimensions due to the fact that the force varies inversely as distance to the power of $(D-2)$. In all dimensions, the centrifugal repulsive force varies inversely as distance cubed. So, in dimensions(> 5), the gravitational pull rises even more rapidly than the centrifugal repulsion. Also, we have observed in our study in sec.II.(A) that an object following timelike geodesic path takes less proper time to reach the singularity in the presence of higher order terms with a positive coupling constant. All these effects seem to make gravitational collapse and spacetime singularities even more likely in higher dimensional and higher order gravity.

The singularity is guaranteed to exist inside the event horizon by virtue of the singularity theorems of Hawking and Penrose [For references and details, see Hawking and Ellis, 1973]. A basic condition for the validity of these theorems is that the timelike convergence condition be satisfied everywhere, namely, $R_{\mu\nu}\ell^{\mu}\ell^{\nu} \geq 0$, where ℓ^{μ} is any timelike or null vector. In GR these theorems remain to be valid provided the gravitational field is coupled only to sources which obey the strong energy condition, and the cosmological constant is either zero or negative. However, if one identifies $S_{\mu\nu}$ in Eq.(I.27) with some sort of stress energy tensor, one can see that this does not satisfy the strong energy condition (but we should point out that such a consideration is not a very comfortable idea since this quantity depends on the geometry itself). We do not yet know the details of the validity of the singularity theorems and black hole uniqueness theorems in such theories.

The effects of the higher order terms on the singularity formation can be properly understood only if one can solve the back reaction problem in such theories. The general procedure for studying back reaction problem in black hole is as follows :

(1) One has to choose a spherically symmetric metric which may be valid inside the black

hole

$$ds^2 = -e^{2p(r)}\,dr^2 + e^{2q(r)}\,dt^2 + r^2\,d\Omega^2. \qquad (II.56)$$

(2) Then one has to consider a field (say, massless scalar field) in such a background and calculate the renormalized vacuum expectation value of the corresponding stress energy tensor denoted by $< T_{\mu\nu} >$. The vacuum state under consideration is the Hartle-Hawking vacuum [Hartle and Hawking, 1976] which describes the blackbody radiation of the scalar massless particles in equilibrium with the thermal bath surrounding the black hole. The temperature at infinity of this bath is $T = S/(2\pi)$.

(3) Then $< T_{\mu\nu} >$ is considered to be the source of the semiclassical Einstein-Gauss-Bonnet equations

$$G_{\mu\nu} - \alpha\kappa S_{\mu\nu} = \kappa < T_{\mu\nu} >, \qquad (II.57)$$

to solve for $p(r)$ and $q(r)$.

Another important quantity for a real scalar field is the renormalized value of the mean square field $< \Phi^2 >$, which may determine the extent of symmetry restoration near a black hole in theories with spontaneous symmetry breaking [Hawking, 1981].

However, a detailed back reaction problem is, in general, notoriously difficult to handle. The procedure described above could never be completed because of the great difficulty in calculating $< T_{\mu\nu} >$. Some approximate calculations leading to one loop quantum correction to the metric in case of the four dimensional Schwarzschild solution ($\alpha = 0$) were, however, done [Howard, 1984; Balbinot and Barlith, 1989]. The problem of vacuum polarization in the gravitational field of a multidimensional black hole was considered by Frolov, Mazzitelli and Paz (1989) for a massless scalar field. One can show that the point splitting method employed by them can be generalised to BDBH as well. In practice, however, the calculation of both $< \Phi^2 >$ and $< T_{\mu\nu} >$ is hampered by the complicated form of the g_{00} component of the metric. Since we could not complete the calculation, we are not going to describe the procedure here, but would like to indicate a few difficulties. The paper by

Frolov *et al* (1989) is quite self explanatory and one may also refer to Birrel and Davies (1982).

The complicated form of 'P' in the BDBH metric poses a great problem for obtaining a solution for the radial part of the Euclidean Green's function $G_E(x, x')$. The solutions for the other parts can be easily obtained. To solve this, one may employ similar techniques used by Iyer, Iyer and Vishveshwara(1989) for the solution of scalar waves in the BDBH background. But the equations become very complicated and getting an analytical solution seems to be an impossible task. The calculation of the Schwinger-DeWitt expansion (SDWE) for the propagator which is very much important for getting the renormalized value also faces various difficulties. For example, we failed to solve an important integral representing geodesic interval $S(\rho, \rho_h) = \int_{\rho_h}^{\rho} P^{-1/2}\, dr$, which is essential for completing this calculation. Another difficulty arises, if one wants to solve the problem in dimensions greater than nine. In that case, one has to know the DeWitt coefficients a_k upto $k = D/2$ for even dimensions and $k = (D-1)/2$ for odd dimensions. However, DeWitt coefficients only upto $k = 4$ are known. Nobody seems to have calculated these coefficients for $k > 4$, which will be very difficult to do.

At the end of this section, we would like to point out an interesting study in the Einstein-Gauss-Bonnet theory, of the collapsing process by Poisson (1991). With suitable boundary conditions, he numerically studied the problem in a black hole spacetime with extra compactified dimensions to

see whether the effect of the higher order terms can reduce the strength of curvature near the singularity.

The boundary conditions considered by him is as follows. Near the singularity, the observed four dimensional slice of the spacetime is assumed to be of the form of Eq.(II.56). The extra dimensions are compactified forming the internal space of the form $w(r)^2 g_{ab} dy^a\, dy^b$. On the other hand, very far away from the singularity, the spacetime is described by the usual four dimensional Schwarzschild metric, $g_{\mu\nu}$, with a constant radius $(D-4)$-torus added to the original line element :

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + w_0^2 [d\chi_1^2 + \cdots + d\chi_{D-4}^2]. \qquad (II.58)$$

The constant radius w_0 is related to the radius of the internal space, $w(r)$, as $w(r) = w_0 \exp[z(r)]$. Then he numerically integrated the field equations inward to solve for $p(r)$, $q(r)$ and $d(r)$. The result shows that for a positive coupling constant α, the singularity occurs sooner than the classical description. For negative coupling the model breaks down because the radius of the internal space becomes zero.

One may attempt to conclude from various results that the extra higher order terms will not come to the rescue of the spacetime near the singularity. But one should remember that the first order quantum corrections should not be considered as the last word in the description of such an important issue like singularity formation. The equations governing the evolution of the spacetime near the singularity may be totally different from what we expect from our present understanding of the subject. Before the final quantum theory is arrived at, the questions related to the singularity theorems in the context of semiclassical gravity, uniqueness theorem and quantum coherence problems are to be rigorously studied.

Chapter III

SEMICLASSICAL DECAY OF THE KALUZA-KLEIN VACUUM

Our aim in this chapter is to describe an important distinctive feature of higher dimensional gravity : semiclassical instability of the ground states and the corresponding decay process. Such a process is fundamentally different from all others in four dimensional gravity where, in fact, it can never occur due to reasons to be described below.

III.(A) Vacuum Decay in Field Theory

Let us first start with a very brief qualitative description of the semiclassical decay processes that arise in ordinary field theories (without gravity). This will help in bringing forth the distinguishing features of the equivalent process in the presence of gravitation when we will describe that later.

The first description of such a process was given by Voloshin, Kobzarev and Okun (1975) and the theory was further developed mainly by Coleman (1977). Consider a self-interacting scalar field Φ in four-dimensional spacetime with nonderivative interactions

$$L = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - U(\Phi). \qquad (III.1)$$

Let U possesses two relative minima, $\Phi_\pm$, only one of which, Φ_- , is an absolute minimum.

The state of the classical field theory for which $\Phi = \Phi_-$ is the unique classical state of the lowest energy and, at least in perturbation theory, corresponds to the unique vacuum state of the quantum theory. The state of the classical field theory for which $\Phi = \Phi_+$ is a stable classical equilibrium state. It is, however, rendered unstable by quantum effects, in particular by barrier penetration. It is a false or metastable vacuum. Once in a while, an energetically favourable bubble of true vacuum will form and this will grow converting the false vacuum to a true one.

The semiclassical process of bubble nucleation can be pictured as the evolution of

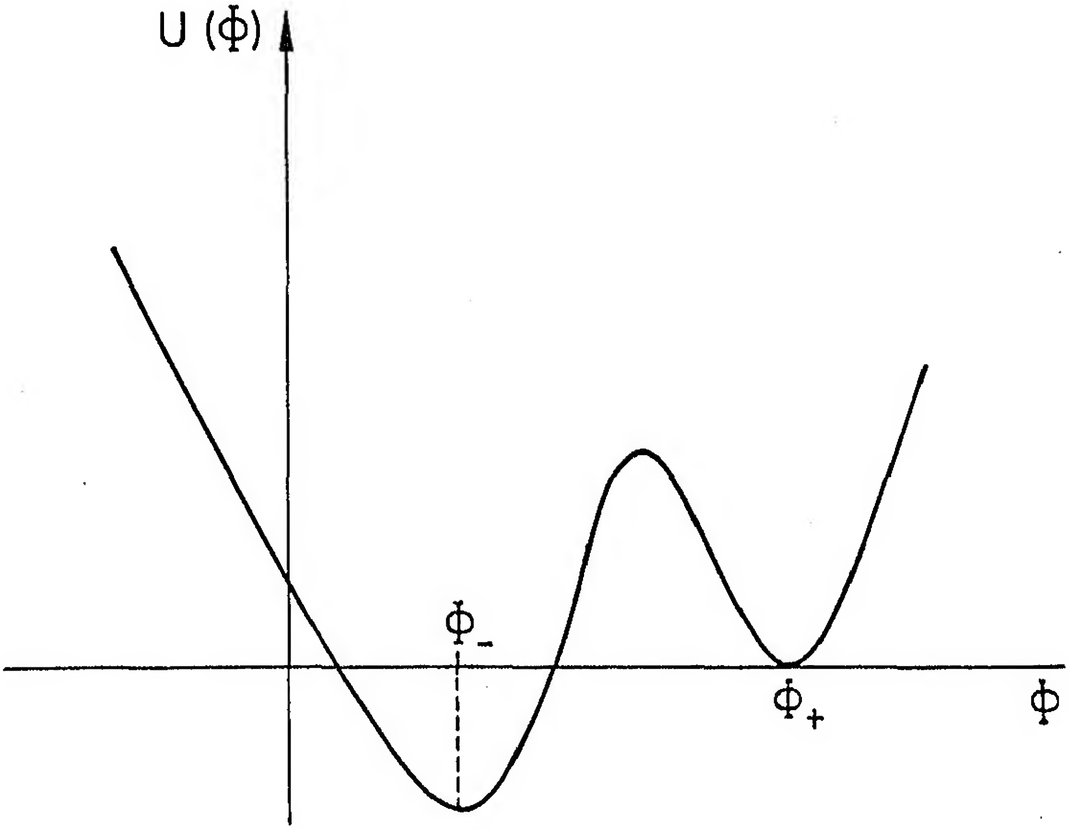

Fig.7 *The assumed shape of the potential of the field* Φ

the field Φ in imaginary time (t_E). To describe the process, therefore, one has to study the corresponding Euclidean field equations. The solution of this equation is called its 'bounce'. This solution approaches the false vacuum value at spacetime infinity and satisfies $\partial\Phi/\partial t_E = 0$ at $t_E = 0$. It can be shown that the field can emerge into the Lorentzian region after the tunneling process only if the eigenvalue spectrum of the small fluctuation operator (essentially the second variation of the action) possesses a negative eigenvalue. Its presence will indicate the instability of the false vacuum.

The probability of bubble nucleation per unit time per unit volume is proportional to $\exp(-I_E)$, where I_E is the Euclidean action for the bounce. The decay process is dominated by the lowest action bounce which has the important property of O(4) invariance, i.e.

$$\Phi(x, t_E) = \tilde{\Phi}(x^2 + t_E^2) \qquad (III.2)$$

To obtain a description of the classical evolution of the bubble after nucleation, one has to analytically continue the bounce solution to the Minkowskian time ($t_E \to it$), so that

$$\Phi(x,t) = \tilde{\Phi}(x^2 - t^2) \tag{III.3}$$

So, the O(4) invariance of the bounce solution implies that the Lorentzian evolution of the bubble is O(3,1) invariant. Equivalently, one can say that the expanding bubble looks the same to all Lorentz observers.

Let us now study, in this context, the guidelines that such a process should follow when we take gravity into account. As will be described below, the situation becomes highly nontrivial and complicated in this case.

III.(B) Positive Energy Theorem and Higher Dimensional Gravity

We recall here that in all reasonable classical field theories, the global energy can be easily expressed as the integral of the local energy density, T_{00}. Since T_{00} is always positive and definite, that naturally ensures the stability of the ground state. However, in gravity, the situation is not so straightforward. In fact, a well-defined concept of local energy density is totally absent in this case. Attempt has been made to realize this by a definition of energy momentum pseudotensor. But first of all, it is not a true tensor and also not positive definite. However, progress along this line has been able to provide a satisfactory definition of the total energy for a gravitating system. The system should, however, be quasi-Minkowskian in nature, so that the metric can be expressed as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$., where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ vanishes at infinity. Then the total energy can be calculated to be in the form of a surface integral :

$$E = \frac{1}{16\pi G} \int \left\{ \frac{\partial h_{ij}}{\partial x^j} - \frac{\partial h_{jj}}{\partial x^i} \right\} dS^i \tag{III.4}$$

The integral is taken over a large surface S^i. This surface integral is popularly known to be ADM(ArnoWitt, Deser, Misner) mass [for details, see sec.7.6, Weinberg, 1972].

It was proved first by Schoen and Yau(1979) using classical methods and then by Witten(1981a) using spinor algebra that this total energy E, in the absence of matter field is always either zero (only for flat Minkowski space) or positive. When matter is present, the statement of the positive energy theorem remains unchanged provided the matter contribution is positive everywhere. We may now summarize all the finer points in the above discussion in a compact statement of the theorem :

Positive Energy Theorem : The total energy of any solution of the four dimensional Einstein equations for which T_{00} of the matter field is either positive or zero at each point in spacetime and in each local Lorentz frame and which asymptotically approaches the flat Minkowski spacetime at infinitely large distances should always be positive or zero, and zero only for flat Minkowski space.

This theorem, therefore, attributes a uniqueness to the gravitational ground state in four dimensions thereby ensuring its semiclassical stability. Unfortunately, the proof of this theorem can not be fully generalised to higher dimensional spacetimes. It is comparatively easier to realise this from the proof forwarded by Witten(1981a). The important steps towards the proof are being described below.

Witten's Proof : Witten's proof crucially depends on the possibility of defining spinors uniquely on an asymptotically Euclidean initial value hypersurface in a gravitating system as shown in Fig.8. Witten begins with the observation that <u>*no*</u> nonzero spinor ϵ that satisfies the Dirac equation $\not{D}\epsilon = \gamma^i D_i \epsilon = 0$ on some initial value hypersurface [the index i denotes spatial coordinates of the hypersurface, γ_i are curved space Dirac Gamma metrices] can, as well, vanish at infinity. He showed that, in the case when matter is present and the Dirac equation is valid, one may write

$$(i \not{D})^2 \epsilon = -\bar{D}^i D_i \epsilon + 4\pi G(T_{00} + T_{0j}\gamma^0\gamma^j)\epsilon = 0. \qquad (III.5)$$

Multiplying by ϵ^* and integrating over the three surface,

$$\int d^3x\sqrt{g}(\bar{D}_i\epsilon^* D^i\epsilon) + 4\pi G \int d^3x\sqrt{g}\epsilon^*[T_{00} + T_{0j}\gamma^0\gamma^j]\epsilon = 0 \qquad (III.6)$$

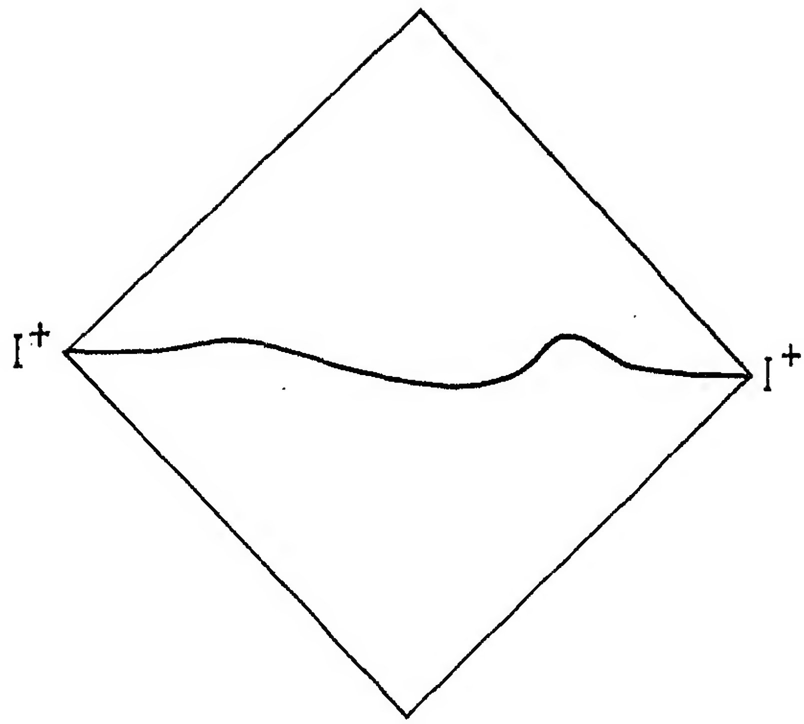

Fig.8 *Asymptotically Euclidean initial value hypersurface*

The surface term vanishes if $\epsilon \to 0$ at infinity. Now, by dominant energy condition which ensures that in any orthonormal basis T_{00} dominates the other components of the energy-momentum tensor, the second term should always be positive (semi-definite). Thus the equation is valid only if the second term is zero (no matter field present) as well as $D_i\epsilon = 0$ which means ϵ =constant. However, if ϵ vanishes at infinity, ϵ should be zero throughout and the above-stated observation follows.

Now, to study spinors that satisfy Dirac's equation but do not vanish at infinity, Witten writes $\epsilon = \epsilon_1 + \epsilon_2$ where ϵ_1 has the asymptotic behaviour at large r of the form $\epsilon_1 = \epsilon_0 + \mathcal{O}(1/r)$; ϵ_0 being a constant and ϵ_2 vanishes at spatial infinity at a rate faster than $\mathcal{O}(1/r)$. He proves that there always exists such an ϵ satisfying Dirac's equation. Then repeating the entire sequence of the previous analysis done for vanishing ϵ, Witten

obtains almost the same result but with an extra surface term that vanished before :

$$\int d^3x \bar{D}_i \epsilon^* D^i \epsilon + 4\pi G \int d^3x\, \epsilon^* (T_{00} + \gamma^0 \gamma^j T_{0j})\epsilon = \int dS^i \epsilon^* D_i \epsilon. \qquad (III.7)$$

dS^i is the area element in a large surface at infinity bounding the three dimensional initial value hypersurface. This surface term can be expressed in terms of the arbritary spinor ϵ_0 and the linearized (or asymptotic) form of the spin *connection* Γ_k. The explicit calculation of S identifies it to be proportional to the ADM mass. Since the L.H.S. of the above equation can never be negative, the global energy should, therefore, always be positive semi-definite.

Unfortunately, this proof cannot be fully generalized to spacetimes with more than four dimensions. Witten's proof applies in any number of dimensions provided the topology of the initial value hypersurface is such that one can consistently define a spinor field on it.

We can now see why the theorem cannot be extended to higher dimensional spacetime with nontrivial topology of initial value hypersurface. We observed that in a spacetime with zero energy, any nonzero spinor must satisfy $\int dS^i \epsilon^* D_i \epsilon = 0$ and, therefore, must be covariantly constant at spatial infinity.

The five dimensional Kaluza-Klein ground state is assumed to be a product of the four dimensionsal Minkowski space and a compactified dimension, $M^4 \times S^1$ and thus represents a multiply connected spacetime. The initial value hypersurface has a topology $R^3 \times S^1$. The presence of the extra compactified dimension introduces a constraint that the phase gained by the spinor on returning to its original value after parallel transport around S^1 must be zero, so that there do not exist inequivalent ways to define spinors.

But if we can find an alternative spacetime with the following properties : (i) with zero energy, (ii) approaches flat $M^4 \times S^1$ spacetime at infinity, (iii) has initial-value hypersurface of different topology that, however, approaches $R^3 \times S^1$ at infinity; then it is possible that the different topology of the hypersurface of this second spacetime would induce a non-zero

phase in ϵ defined on $M^4 \times S^1$ flat spacetime. So, a consistent spinor field that will be covariantly constant at spatial infinity can never be constructed and we will not be able to apply Positive energy theorem in such cases. It is quite possible that in this case the ground state may decay into another spacetime of same or lower energy. That is just the way by which Witten proved the semiclassical instability of $M^4 \times S^1$ by finding an alternative spacetime (known as Witten Bubble spacetime) that possesses all the three properties above.

But before getting into that, let us sum up here the general technical procedure that is to be followed up to study the semiclassical instability of the ground state of any higher dimensional theory of gravitation. Since the ground state corresponds to zero energy, as discussed above, it can decay into another spacetime of zero energy only. So, the existence of the alternative spacetime will essentially disprove the uniqueness of the vacuum. The procedure for obtaining the alternative solution is as follows :

(1) Try to get a solution of the Euclidean field equations such that it approaches the analytically continued Euclidean version of the assumed ground state at infinity. This solution is the instanton-like 'bounce' solution which interpolates between the assumed vacuum and whatever spacetime it decays into.

(2) Search for the negative action modes in the functional determinant obtained for small fluctuation around this 'bounce' solution. If such modes exist, the gaussian integral around this solution will contribute an imaginary part to the energy of the vacuum state, thus representing the instability [for details, see Gross, Perry, Yaffe,1982].

(3) To obtain the spacetime into which the assumed ground state decays, analytically continue the 'bounce' solution back to the Minkowski space. If it remains to be real-valued metric there, then that will represent the alternative spacetime.

In the next section we are going to discuss how Witten found the bubble solution.

III.(C) The Witten Bubble Solution

The Euclideanised version of the Kaluza-Klein ground state has topology $R^4 \times S^1$ which has an asymptotically (in fact, everywhere) $S^3 \times S^1$ boundary.

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 + dt^2 + d\chi^2 \\ &= dr^2 + r^2 d\bar{\Omega}_3^2 + d\chi^2 \end{aligned} \tag{III.8}$$

where χ represents the compactified dimension. To search for the bounce solution, Witten realised that the five dimensional Euclidean Schwarzschild solution

$$ds^2 = \left(1 - \frac{2GM}{r^2}\right)dt_E^2 + \left(1 - \frac{2GM}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \tag{III.9}$$

also has an $S^3 \times S^1$ boundary with an asymptotically flat metric. One may write it in a somewhat different way as

$$ds^2 = \left(1 - \frac{\alpha}{r^2}\right)^{-1} dr^2 + \left(1 - \frac{\alpha}{r^2}\right)d\chi^2 + r^2 d\Omega_3^2, \tag{III.10}$$

so that it still remains to be a solution of the Euclideanised field equations. The quantity α should not be interpreted to be equal to $2GM$ here. Rather it is to be considered as a parameter. Now, studying the behaviour of this metric at $r = \sqrt{\alpha}$, he found that χ has to be periodic with a period $2\pi\sqrt{\alpha}$ so that the r–χ subspace remains nonsingular. Thus, $\alpha = R_0^2$ where R_0 is the radius of the fifth dimension, a completely free parameter of the theory.

Since the boundary conditions of Eqs. III.8 and 10 are now same, (III.10) can, therefore, represent the Kaluza-Klein instanton. Also, Euclidean Schwarzschild solution has one transverse traceless negative mode for small oscillations. So, the one loop determinant is imaginary representing the decay of flat space.

The instanton has a discrete Z_2 time symmetry. Thus, it also has a surface ($t_E = 0$) on which the time derivative of the metric vanishes, or stated more geometrically, the extrinsic curvature vanishes. So, after the analytical continuation ($t_E \to \pi/2 + i\tau$) the Minkowski solution is obtained as

$$ds^2 = -r^2 d\tau^2 + \left[1 - \frac{R_0^2}{r^2}\right]^{-1} dr^2 + r^2 \cosh^2\tau(d\theta^2 + \sin^2\theta d\phi^2) + \left[1 - \frac{R_0^2}{r^2}\right] d\chi^2 \tag{III.11}$$

where r has the range $R_0 \leq r < \infty$.

The topology of the $\tau = 0$ surface is $R^2 \times S^2$, although in its geometry it is asymptotic to flat metric on $R^3 \times S^1$. As described in the previous section, this solution may thus induce a nonzero phase in a spinor (upon parallel transport) defined on $R^3 \times S^1$ in Kaluza-Klein vacuum and become a cause for the semiclassical instability.

The instanton represents a tunneling from flat $R^3 \times S^1$ to $R^2 \times S^2$ and thus involves a topology change. This is one of the rarest examples of the topology changing processes in gravitation [see Strominger, 1989].

The spacetime (III.11) is known as Witten Bubble. Its evolution properties are being described in the next section.

III.(D) Evolution of The Witten Bubble

Let us first introduce here 'spherical Rindler' coordinates to describe the four dimensional Minkowskian subspace of the Kaluza-Klein ground state. A spherical array of uniformly accelerated observers uses such type of 'hyperbolic' coordinates. These are related to the Minkowskian coordinates in the following way :

$$\begin{aligned} t &= r \sinh \tau, \\ x^1 &= r \cosh \tau \cos \phi \sin \theta, \\ x^2 &= r \cosh \tau \sin \phi \sin \theta, \\ x^3 &= r \cosh \tau \cos \theta. \end{aligned} \tag{III.12}$$

Here, we are using these coordinates to make the vacuum metric comparable with the metric of the Witten bubble. The five dimensional Kaluza-Klein vacuum metric can now be written as

$$ds^2 = -r^2 d\tau^2 + dr^2 + r^2 \cosh^2 \tau (d\theta^2 + \sin^2 \theta d\phi^2) + d\chi^2. \tag{III.13}$$

This metric is valid for $\tau < 0$. For $\tau > 0$, the decay state of the Kaluza-Klein vacuum has to be described by the Witten Bubble metric (III.11).

As a result of this decay, a microscopic hole of radius R_0 will be spontaneously formed in space. Like the bubble wall in conventional vacuum decay, this hole will start expanding to infinity with a uniform acceleration (Fig.9) and, therefore, will approach the velocity of light. The evolution of the Witten bubble is also $O(3,1)$ invariant and looks to be same to all lorentzian observers. But we should emphasize here that the range of r ($R_0 \leq r < \infty$) actually represents the fact that, unlike conventional decay where the inside of a bubble corresponds to the true vacuum, the Witten bubble has no interior at all. The Physical spacetime corresponds to the bubble wall and its exterior only.

Also, we cannot call the bubble surface to be a 'horizon' because, unlike 'horizons' of black hole or Rindler system, the hyperbola corresponding to the wall represents the termination of the spacetime itself and no information can be received from or sent to any other region. As it has already been discussed in the previous section, the 'troublesome part' of the metric or the $r - -\chi$ subspace reduces to a planar surface at $r = R_0$. This implies that the manifold is 'smooth' or geodesically complete there. The $r - -\chi$ subspace asymptotically approaches the line element of a cylinder.

The bubble surface is a 2-sphere of area $4\pi R_0^2 \cosh^2 \tau$. So, at any particular instant t, its radius is $r(t) = \sqrt{R_0^2 + t^2}$. For very large r, the metric (III.11) asymptotically approaches the $M^4 \times S^1$ spacetime described by Eq.III.13.

These interesting properties of the Witten bubble have also been verified by the study of both time-like and null geodesics by Brill and Matlin (1989). In the next chapter, we shall study scalar waves in the Witten Bubble background. This will reveal some more interesting features of the evolution of such a spacetime.

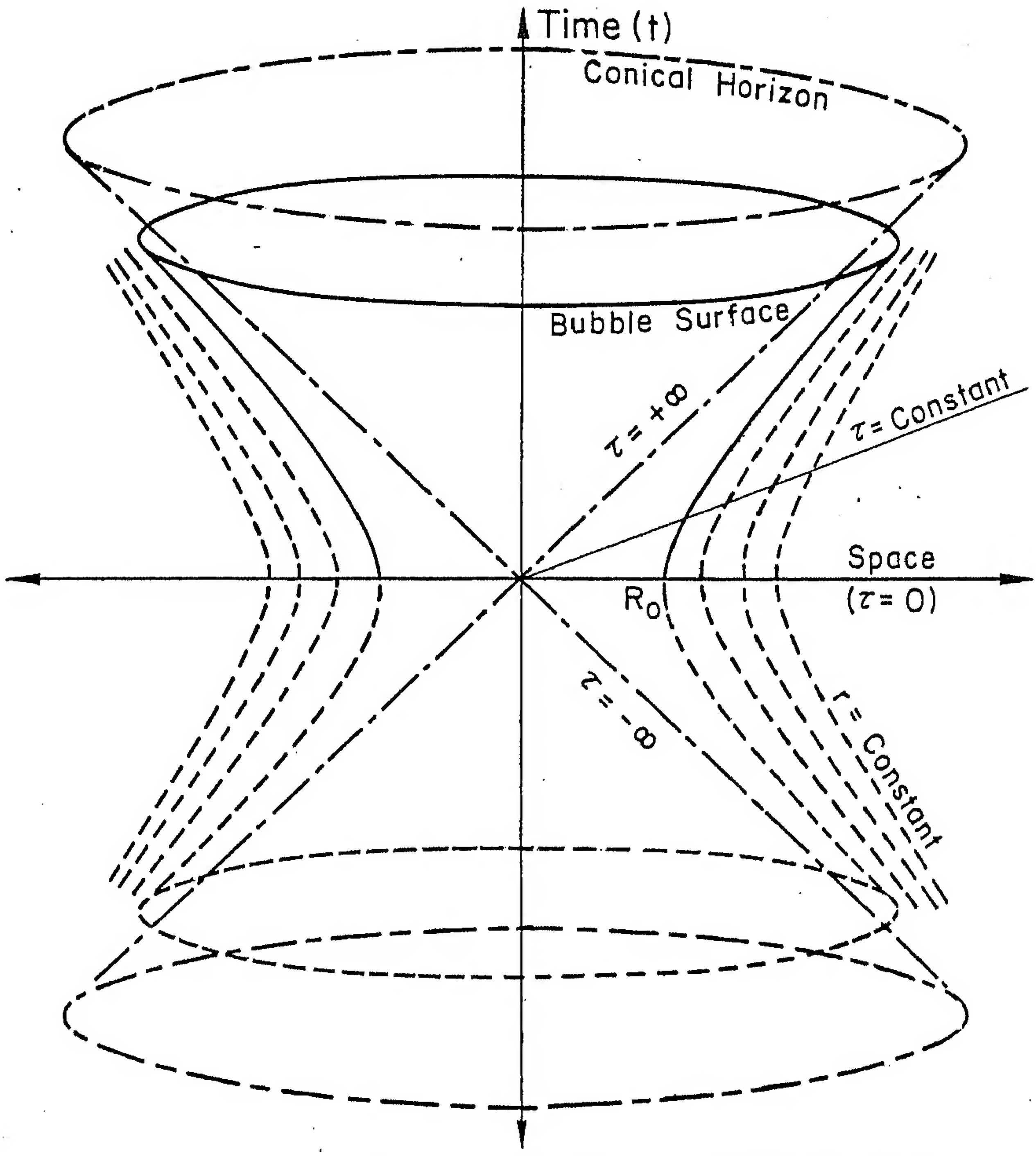

Fig.9 *Evolution of Bubble in a 2+1 dimensional Minkowski subspace. Radial lines represent constant τ lines, whereas any hyperbola corresponds to a constant value of r. The bubble is formed at $\tau = 0$.*

Chapter IV

SCALAR WAVES IN THE WITTEN BUBBLE BACKGROUND

Different classical properties of the Witten Bubble spacetime introduced in the last chapter have been studied in detail by Matlin(1991). He also investigated different semi-classical phenomena, e.g. particle production, back reaction problem etc., involved in the process of the formation and evolution of the bubble.

The nature of the time-like and null geodesics in this spacetime has been studied by Brill and Matlin (1989). As in the case of the geodesics, the study of the behaviour of scalar waves also probes the geometry of the spacetime. The scattering phenomenon throws light on the nature of the bubble as well as its effect on the surrounding spacetime. Further, it provides us valuable information about the bound states and the stability of the spacetime. Also, the investigation of scalar waves in an exact solution such as the Witten metric offers insight into the propagation of waves in strong gravitational fields.

In this chapter, we describe in detail all these topics related to scalar waves in the Witten bubble [Bhawal and Vishveshwara, 1990]. Associated with this chapter are Appendices B and C – the former contains alternative scalar wave solutions in the Witten bubble metric and the latter is a discussion on some coordinate transformation that we use in this chapter.

IV.(A) The Klein-Gordon Equation

The Klein-Gordon equation for a massive scalar field is given by

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\mu\nu}\Phi_{,\nu})_{,\mu} - M^2\Phi = 0 \qquad (IV.1)$$

where M is the mass of the field.

The metric (III.11) is independent of the fifth coordinate χ and, therefore, there is a Killing symmetry in the fifth dimension. The solution of the Klein-Gordon equation is

found to be

$$\Phi = \mathcal{R}(r) H^{\ell m}_{i\omega}(\tau, \theta, \phi)\, e^{i m_1 \chi} \tag{IV.2}$$

$$= \mathcal{R}(r)\, T^{\ell}_{i\omega}(\tau)\, Y^{m}_{\ell}(\theta, \phi)\, e^{i m_1 \chi}, \tag{IV.3}$$

where $Y_l^m(\theta\phi)$ are the spherical harmonics. Other functions appearing in this solution will be discussed in the following subsections.

Scalar waves for which the fifth dimensional component vanishes ($m_1 = 0$) represent the propagation of ordinary scalar waves in such a spacetime. The case of nonzero m_1 cannot readily be interpreted in terms of realistic scalar waves [see Bailin and Love, 1987]. Since the standard studies in five dimensional Kaluza-Klein theory take the compactified radius of the extra dimension to be of the order of the Planck length, $L_{\rm Pl}$, a mode characterised by the quantum number m_1 then corresponds to a wavelength along the fifth dimension of the order of $2\pi L_{\rm Pl}/m_1$. The corresponding momentum or the relativistic kinetic energy then turns out to be of order $m_1.M_{\rm Pl}$ (in units $c = G = \hbar = 1$), where $M_{\rm Pl}$ is the Planck mass ($\simeq 10^{19}$ GeV). Such highly energetic waves do not represent realistic ones encountered in our observed physical environment.

So, in this work, we consider $m_1 = 0$ throughout. We shall see that this will greatly facilitate the solution of the radial equation in such a spacetime.

(a) *τ-Equation*

The wave-field represented by the solution (IV.2) does not oscillate in a simple harmonic way, but in a more complicated way given by the hyperbolic harmonics $H^{\ell m}_{i\omega}(\tau, \theta, \phi)$. These hyperbolic harmonics are characterised by the generalised frequency ω, which labels the representation of the Lorentz group SO(3,1). These are actually the eigenfunctions of the D'Alembertian on the unit time-like hyperboloid :

$$[-\frac{1}{\cosh^2\tau}\frac{\partial}{\partial\tau}\cosh^2\tau\frac{\partial}{\partial\tau} + \frac{1}{\cosh^2\tau}\nabla^2_{\theta,\phi}]H^{\ell m}_{i\omega} = (\omega^2 + 1)H^{\ell m}_{i\omega}. \tag{IV.4}$$

These form a complete orthonormal set with respect to the Lorentz-invariant volume

on the time-like hyperboloid :

$$\int_0^\infty \int_0^{2\pi} \int_0^\pi H_{i\omega}^{\ell m}(\tau,\theta,\phi)\, H_{i\omega'}^{*\ell' m'}(\tau,\theta,\phi)\, \cosh^2\tau \sin\theta \, d\theta d\phi d\tau = \delta(\omega-\omega')\delta_{\ell\ell'}\delta_{mm'}. \tag{IV.5}$$

A detailed construction of these hyperbolic harmonics has been discussed by Gerlach(1983) in the appendix of his paper. Some inadvertant errors seem to have crept into his constructions, probably stemming from the original sources used. In this work, we use explicit expressions for the solutions, thereby avoiding any possible ambiguities.

$$\text{Also,} \quad H_{i\omega}^{\ell m}(\tau,\theta,\phi) = T_{i\omega}^{\ell}(\tau) Y_\ell^m(\theta,\phi) \tag{IV.6}$$

where $T_{i\omega}^{\ell}(\tau)$ satisfies the equation

$$[-\frac{1}{\cosh^2\tau}\frac{d}{d\tau}\cosh^2\tau\frac{d}{d\tau} - \frac{\ell(\ell+1)}{\cosh^2\tau}]T_{i\omega}^{\ell}(\tau) = (\omega^2+1)T_{i\omega}^{\ell}(\tau) \tag{IV.7}$$

Introducing the function

$$u_{i\omega}^{\ell}(\tau) = \cosh\tau T_{i\omega}^{\ell}(\tau), \tag{IV.8}$$

we can write Eq.(IV.7) in the Schrödinger-form

$$[-\frac{d^2}{d\tau^2} - \frac{\ell(\ell+1)}{\cosh^2\tau}]u_{i\omega}^{\ell}(\tau) = \omega^2 u_{i\omega}^{\ell}(\tau)$$

In this section, we are confining our discussion only to the lowest mode ($\ell = 0$). The higher mode solutions ($\ell = 0$) will be discussed in section IV.(C).

The lowest mode solution is given by

$$u_{i\omega}^{0}(\tau) = \frac{e^{i\omega\tau}}{\sqrt{4\pi}} \tag{IV.10}$$

where $1/\sqrt{4\pi}$ factor has been taken to normalise the function.

$$\int_0^\infty u_{i\omega}^{0}(\tau)u_{i\omega'}^{0}(\tau)d\tau = \delta(\omega-\omega'). \tag{IV.11}$$

$$\text{Therefore,} \quad H^{0m}_{i\omega}(\tau,\theta,\phi) = \frac{1}{\sqrt{4\pi}} \frac{e^{i\omega\tau}}{\cosh\tau} e^{im\phi} \tag{IV.12}$$

An alternative solution for τ-equation (IV.7) is discussed in Appendix A.

(b) *Radial equation*

The radial function $\mathcal{R}(r)$ satisfies the following equation :

$$\begin{aligned} 0 =& \frac{1}{r^3}\left[1 - \frac{R^2}{r^2}\right]\left[r^3(1 - \frac{R_0^2}{r^2})\mathcal{R}_{,r}\right]_{,r} - M^2\left[1 - \frac{R_0^2}{r^2}\right]\mathcal{R} \\ & - m_1^2\mathcal{R} + \frac{\omega^2+1}{r^2}\left[1 - \frac{R_0^2}{r^2}\right]\mathcal{R} \end{aligned} \tag{IV.13}$$

As discussed in the beginning of this section, we shall take $m_1 = 0$. Then for massless ($M = 0$) scalar waves, Eq.(IV.13) turns out to be

$$\frac{1}{r}[r(r^2 - R_0^2)\mathcal{R}_{,r}]_{,r} + (\omega^2+1)\mathcal{R} = 0 \tag{IV.14}$$

Now, we make a change of variable such that

$$\frac{dr}{dx} = \sqrt{r^2 - R_0^2} \tag{IV.15}$$

$$\text{or,} \qquad x = \cosh^{-1}\left(\frac{r}{R_0}\right) \tag{IV.16}$$

As $r \to R_0$, $x \to 0$. As $r \to +\infty$, x goes to both $\pm\infty$. Here, we are choosing the limit to be $x \to \infty$. Use of such a coordinate transformation has a natural significance which we shall discuss in Appendix C.

After this transformation, Eq.(IV.14) becomes

$$\mathcal{R}_{,x,x} + f(r)\mathcal{R}_{,x} + (\omega^2+1)\mathcal{R} = 0 \tag{IV.17}$$

$$\text{where} \qquad f(r) = \frac{r}{\sqrt{r^2 - R_0^2}} + \frac{\sqrt{r^2 - R_0^2}}{r}. \tag{IV.18}$$

The radial equation is still not free of first derivative terms. Now, if we define

$$\mathcal{R}(r) = \frac{\Psi(r)}{\sqrt{r\sqrt{r^2 - R_0^2}}} \tag{IV.19}$$

then it brings Eq.(IV.17) to the form of a Schrödinger equation :

$$\Psi_{,x,x} + \left[\omega^2 + \frac{1}{\sinh^2 2x}\right]\Psi = 0 \tag{IV.20}$$

with an effective potential

$$V_{\rm eff} = -\frac{1}{\sinh^2 2x} \tag{IV.21}$$

This Schrödinger equation has a very simple effective potential whose behaviour can be readily visualised. Qualitative features of the wave behaviour can also be easily discussed. To solve this equation, let us introduce a new variable

$$y = -\sinh^2 2x \tag{IV.22}$$

so that Eq.(IV.20) becomes

$$y(1-y)\Psi_{,y,y} + (\frac{1}{2} - y)\Psi_{,y} + \left(\frac{1}{16y} - \frac{\omega^2}{16}\right)\Psi = 0. \tag{IV.23}$$

Then we define a new function

$$W = y^{-1/4}\Psi \tag{IV.24}$$

which now satisfies the following equation

$$y(1-y)W_{,y,y} + [1 - \frac{3}{2}y]W_{,y} - \frac{\omega^2+1}{16}W = 0. \tag{IV.25}$$

This is in the form of a hypergeometric equation. Therefore, the analytical solutions of this equation near $y = 0$ can be found in terms of hypergeometric series. These are

$$W_1 = F(a, b; 1; y) \tag{IV.26}$$

$$\text{and} \quad W_2 = \ln y F(a, b; 1; y) + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} y^n S(n) \tag{IV.27}$$

for $|\, y \,| < 1$, where

$$\begin{aligned}
a &= \frac{1}{4}(1 + i\omega), \\
b &= \frac{1}{4}(1 - i\omega), \\
F(a, b; 1; y) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} y^n, \\
(a)_n &= \frac{\Gamma(a+n)}{\Gamma(a)}, \\
S(n) &= \psi(a+n) - \psi(a) + \psi(b+n) - \psi(b) - 2\psi(n+1) + 2\psi(1),
\end{aligned}$$

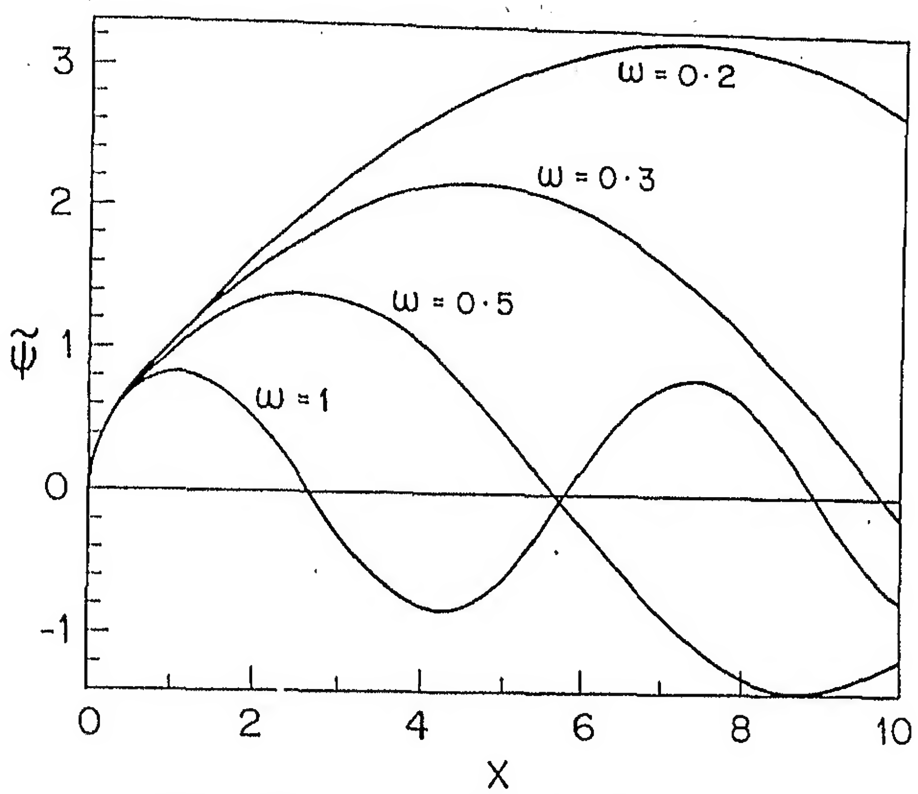

Fig.10 *The solution $\Psi_1(x)$ for different frequencies ω*

where ψ is the logarithmic derivative of the Gamma function.

Using Eq.(IV.24), we can now get the solutions of the Schrödinger equation (IV.20) to be

$$\Psi_1 = y^{1/4} F(a, b; 1; y) \qquad (IV.28)$$

$$\Psi_2 = y^{1/4} \left[\ln y F(a, b; 1; y) + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} y^n S(n) \right] \qquad (IV.29)$$

Both these solutions go to zero, as $y \to 0$.

We have also solved the Eq.(IV.20) numerically and plotted the solutions in figure 10 for different frequencies ω.

We observe that starting from $x = 0$, the solution rises very rapidly to a maximum value and then starts oscillating like a cosine wave. As ω increases, the influence of the

spacetime on the waves reduces and the solution starts oscillating very close to the bubble wall.

However, if we look at the corresponding solutions for $\mathcal{R}$-equation by using Eq.(IV.19)

$$\mathcal{R}_1 = \frac{1+i}{R_0} F(a,b;1;y) \tag{IV.30}$$

$$\mathcal{R}_2 = \frac{1+i}{R_0}\left[\ln y\, F(a,b;1;y) + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} y^n S(n)\right] \tag{IV.31}$$

We observe that at $r = R_0$,

$$\begin{aligned} \mathcal{R}_1 &= \frac{1+i}{R_0}\,. \\ \text{whereas} \qquad \mathcal{R}_2 &\rightarrow -\infty \end{aligned} \tag{IV.32}$$

Therefore, as far as the Schrödinger equation is concerned, the second solution behaves properly in that coordinate system. But when we consider the actual radial equation, the corresponding solution blows up at $r = R_0$. So, we are discarding the second solution throughout our further calculations.

From Eq.(IV.20), we can readily obtain the asymptotic behaviour of its solution as $x \rightarrow \infty$

$$\Psi(x \rightarrow +\infty) = Ae^{-i\omega x} + Be^{+i\omega x} \tag{IV.33}$$

where A and B are arbitrary constants. Then, using Eq.(IV.19), we get

$$\mathcal{R}(r \rightarrow +\infty) = \frac{Ae^{-i\omega x} + Be^{+i\omega x}}{r}, \tag{IV.34}$$

since as $r \rightarrow +\infty$, $\sqrt{r\sqrt{r^2 - R_0^2}} \rightarrow r$.

We shall now apply these considerations to the wave scattering by the bubble.

IV.(B) Scattering and Bound States

The total scalar wave solution in its lowest mode can now be written in its asymptotic limit to be

$$\Phi(m_1, \ell = 0, r \to +\infty) = \frac{e^{im\phi}}{\sqrt{4\pi}} \frac{e^{i\omega\tau}}{r\cosh\tau}(Ae^{-i\omega x} + Be^{+i\omega x}) \qquad (IV.35)$$

The factor $(r\cosh\tau)$ in the denominator ensures that the total flux of energy passing through a unit solid angle $d\Omega$ does not depend on r or τ for very large r.

What is the relation between A and B ? The answer follows immediately, if we just consider the behaviour of the differential equation (IV.17). The hypergeometric series of Eq.(IV.30) is always real, since a and b are complex conjugates of each other. Now, if we use initial condition (IV.32) in Eq.(IV.17) and study the evolution of $\mathcal{R}$, we shall see that the real and imaginary parts of this equation will evolve independently. However, at any point, both these parts will be equal. Considering this fact and matching the solution with (IV.34) in the asymptotic limit, one can show by a very simple calculation that this is a case which corresponds to $\mid A \mid = \mid B \mid$.

The actual expressions for 'A' and 'B' can be obtained by analytically extending the solution (IV.28) to infinitely large negative values of the argument

$$y = -\sinh^2 2x \to -2^{-2}e^{4x}. \qquad (IV.36)$$

Then the solution is

$$\begin{aligned} \Psi(x \to +\infty) =& 2^{-1/2}e^{x}\left[\frac{\Gamma(b-a)}{\Gamma(b)\Gamma(1-a)}2^{2a}e^{-4ax} + \frac{\Gamma(a-b)}{\Gamma(a)\Gamma(1-b)}2^{2b}e^{-4bx}\right] \\ =& \frac{\Gamma(-i\frac{\omega}{2})e^{i\frac{\omega}{2}\ln 2}}{\Gamma(\frac{1}{4}-i\frac{\omega}{4})\Gamma(\frac{3}{4}-i\frac{\omega}{4})}e^{-i\omega x} + \frac{\Gamma(+i\frac{\omega}{2})e^{-i\frac{\omega}{2}\ln 2}}{\Gamma(\frac{1}{4}+i\frac{\omega}{4})\Gamma(\frac{3}{4}+i\frac{\omega}{4})}e^{+i\omega x}. \end{aligned} \qquad (IV.37)$$

Matching with Eq.(IV.33), we get expressions for A and B. Since $\Gamma(\bar{z}) = \bar{\Gamma}(z)$, we can easily see that A and B are complex conjugates of each other and, therefore, $\mid A \mid = \mid B \mid$.

From the foregoing discussion, we see that only one of the two independent solutions is acceptable. This solution is well-behaved at infinity and consists of incoming and reflected

wave components with equal amplitudes. Further, this solution goes to zero at $r = R_0$. Since the other solution is not well-behaved at $r = R_0$, there is no scope for superposition of the two solutions, thereby obtaining other boundary conditions, e.g., standing waves that do not go to zero at $r = R_0$.

On the other hand, the boundary conditions that have naturally arisen fit in well with the notion of a bubble surface enclosing a region $r < R_0$ that does not correspond to points in physical space. One expects the incoming wave to be totally reflected from the bubble surface. This phenomenon is, in fact, happening here. We may also note here that by a similar argument, one can rule out quasinormal modes of the bubble, since waves purely incoming at $r = R_0$ and purely outgoing at $r \to \infty$ cannot be obtained. This indicates that the bubble surface acts as a perfectly reflecting rigid barrier.

To investigate the bound states of this problem, we have to consider imaginary frequencies. Let us replace $i\omega \to \omega_n$. Then, for τ-equation (IV.9), a discrete set of square-integrable wave functions can be obtained as bound states. These have been constructed in detail by Gerlach(1983).

To obtain bound states in the radial equation(IV.20), we see that the parameters a and b in solution (IV.28) have now become real. Then, in the asymptotic expressions (IV.37), the first term behaves as $e^{-\omega_n x}$ and the second as $e^{+\omega_n x}$. Bound states are possible, only if the coefficient of $e^{+\omega_n x}$ in the second term vanishes. However, all Γ functions in this coefficient have a positive real argument. Therefore, no Γ function in the denominator can ever blow up and make the factor vanish. Consequently, no bound state is possible.

Nevertheless, we should point out here that if one performs the following integration

$$\int_0^\infty \sqrt{E - V_{\text{eff}}}\, dx$$

for $E = 0$ in this case, one obtains

$$\int_0^\infty \frac{1}{\sinh 2x}\, dx = \frac{1}{2} \ln \tanh x \Big|_0^{+\infty} = +\infty.$$

Following Merzbacher (1970), this means the existence of an infinite number of bound states. However, our explicit calculation has shown that there is no bound state at all. This apparent contradiction is due to our discard of solution (IV.29), though it was behaving well throughout the range of variable x in Schrödinger equation (IV.20). Had we considered both solutions, we would have obtained an infinite number of bound states. But those are not realistic as far as our problem is concerned.

Now, since the same ω appears in both radial and τ equations, the nonexistence of bound states also confirms that modes exponentially growing with τ do not exist. This shows the mode stability of the bubble spacetime against scalar perturbations. Further, since the scattering modes form a complete set, the bubble spacetime is stable with respect to any arbitrary scalar perturbations.

IV.(C) Higher Mode ($\ell > 0$) Solutions

As we have seen in section IV.(A), the lowest mode ($\ell = 0$) solution given in Eq.(IV.10) is δ-function normalised. Now, to study higher mode solution, following Gerlach(1983), we can introduce the raising and lowering operators by factorization method in Eq.(IV.9),

$$L_{\pm}^{\ell} = \ell \tanh \tau \mp d\tau \qquad (IV.38)$$

Then one can write $u_{i\omega}^{\ell}(\tau)$ as an eigenfunction of $L_{+}^{\ell} L_{-}^{\ell}$, with the eigenvalue $(\omega^2 + \ell^2)$. Now the general eigenfunction can be written in its normalised form to be

$$u_{i\omega}^{\ell}(\tau) = \left[(\omega^2 + \ell^2) \cdots (\omega^2 + 1^2)\right]^{-1/2} L_{+}^{\ell} \cdots L_{+}^{1} \frac{e^{i\omega\tau}}{\sqrt{4\pi}} \qquad (IV.39)$$

$$\text{Also,} \quad u_{i\omega}^{\ell+1}(\tau) = \left[\omega^2 + (\ell+1)^2\right]^{-1/2} L_{+}^{\ell+1} u_{i\omega}^{\ell}(\tau). \qquad (IV.40)$$

For $\ell = 1$, we obtain from Eq.(IV.39),

$$\begin{aligned} u_{i\omega}^{1}(\tau) &= (\omega^2 + 1)^{-1/2} (\tanh \tau - \frac{d}{d\tau}) \frac{e^{i\omega\tau}}{\sqrt{4\pi}} \\ &= \mathcal{A}_1(\tau) e^{-i\Theta_1(\tau)} \frac{e^{i\omega\tau}}{\sqrt{4\pi}} \end{aligned} \qquad (IV.41)$$

IV.(D) Concluding Remarks

In the previous sections, we have developed the mathematical formalism for and studied the behaviour of scalar field in the Witten Bubble spacetime. We have written the eigenfunctions of the temporal equation as hyperbolic harmonics which manifest wave-behaviour in all of its modes. By choosing the null coordinate system, we could transform the radial equation into a very simple Schrödinger form. Studying the scattering problem, we have observed that our results are consistent with the concept of bubble as a perfectly reflecting wall. At large enough distance, we could get both incoming and outgoing waves with the same amplitude, thus giving the value of the reflection to be unity. On the other hand, near the bubble, the wave behaviour gets distorted. The higher the frequency, the lower is the distortion produced by the spacetime. A high frequency wave starts manifesting its wave behaviour very near to the bubble wall.

A study of bound states confirms the stability of the spacetime against arbitrary scalar perturbations. For a complete stability analysis of such a spacetime, the study of electromagnetic and gravitational perturbations is also necessary. This study may be able to project a clearer concept of some inherent aspects of the Witten bubble and lead to further studies related to such a spacetime.

Chapter V

SEMICLASSICAL DECAY OF THE KALUZA-KLEIN VACUUM IN HIGHER ORDER GRAVITY

In this chapter, we are going to extend the ideas and techniques stated in chapter III to the vacuum state coming from theories involving higher order terms. The starting point is the classical D-dimensional Einstein action modified by the Gauss-Bonnet combination of higher order terms given by Eq.(I.26).

As has been discussed in chapter III, a very much related question that one has to address in this regard is of the validity of the Positive Energy Theorem (PET) in this theory. There are exactly two distinct considerations :

(i) effect of the presence of the extra higher order terms,

(ii) the topology of the gravitating system.

The first consideration in an almost similar situation (i.e. $R + R^2$ gravity in four dimensions) was investigated by Strominger (1984) who proved PET in that theory. Also, flat space was shown to be the unique topologically Minkowskian stationary point of energy.

It is not too obvious that the PET will remain to be valid in the Einstein- Gauss-Bonnet theory or the Lovelock gravity as a whole. In field equations (I.27) if one identifies the contribution of the Gauss-Bonnet term with some stress energy tensor, $S_{\mu\nu}$ (although it is not a very comfortable idea), one may see that S_{00} is of indefinite sign. So, although the energy consideration is dependent on the dynamics at large distances which is mainly determined by the Einstein term, a negative sign of S_{00} (for $\alpha > 0$) may make the total energy negative.

To see whether Witten-type arguments can be applied in this case, one has to take into account supersymmetric considerations. It seems plausible that the model incorporating

Gauss-Bonnet term has a supersymmetric extension at least for $\alpha > 0$ [Deser, 1986]. It was also pointed out by Boulware and Deser (1985) that, due to this reason, the global energy will be positive only for the asymptotically Minkowskian flat solutions (a consideration also used in the proof of the theorem in four dimensions, see sec. III.B), but not for those which asymptotically approach de Sitter or anti de-Sitter spacetime.

As far as the consideration (ii) regarding the topology of the system is concerned, all arguments presented in sec. III(B) can be extended to this case. Here, we show that the theorem will not be valid for a multiply connected topology of the initial value hypersurface.

We consider $M^4 \times S^1$ to be the flat spacetime solution of such a theory. We are interested in the case of pure gravity without matter fields and, therefore, we study the source free Einstein-Gauss-Bonnet field equations given by Eq.(I.28) [Bhawal and Mani, 1992].

Due to such a choice of the topology of the ground state, the 5-dimensional gravitational constant is taken to be $G_5 = 2\pi G R_0$, where R_0 is the radius of the fifth dimension. So, by definition, $\kappa = 32\pi^2 G R_0$.

Now, to look for a semiclassical instability of this vacuum state, we have to search for an instanton-like 'bounce' solution of the classical Euclidean field equations of the higher order theory. Such a solution should be asymptotic to the flat infinite Euclidean spacetime

$$ds^2 = dx^2 + dy^2 + dz^2 + dt^2 + d\chi^2 \qquad (V.1)$$

By introducing polar coordinates $(\rho, \psi, \theta, \phi)$ in the four noncompact dimensions x,y,z ,t, we can rewrite this as

$$ds^2 = d\rho^2 + \rho^2[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)] + d\chi^2 \qquad (V.2)$$

Also, this solution will represent an instability of the vacuum, if there exists negative action modes in small fluctuations around it. To look for such a solution, we can use the five-dimensional Boulware-Deser metric [Eq.II.1 & 5]. After analytically continuing that

metric to the Euclidean time, the solution can be written as (in dimensionless variables):

$$ds^2 = P^{-1}d\rho^2 + \rho^2[d\psi^2 + sin^2\psi(d\theta^2 + sin^2\theta d\phi^2)] + Pd\chi^2 \qquad (V.3)$$

$$P = 1 + \frac{\rho^2}{2a}[1 - (1 + \frac{8am}{\rho^4})^{1/2}], \qquad (V.4)$$

where $a = 32\pi\alpha$ (see Eq.II.4). The quantity m is an arbitrary parameter of dimension $[L^2]$. This solution satisfies the Euclidean field equations for any value of m, except at $\rho = \rho_i = \sqrt{2m - a}$, where P becomes zero. To study the behaviour at ρ_i, we make a coordinate transformation $\rho = \rho_i + \lambda^2$, where λ^2 is very small. Then as $\lambda^2 \to 0$, the $\rho - \chi$ subspace will behave as

$$\frac{2(2m + a)}{\sqrt{2m - a}}\left[d\lambda^2 + \lambda^2\frac{2m - a}{(2m + a)^2}d\chi^2\right] \qquad (V.5)$$

where we have neglected the higher order terms of (λ^2) in $g_{\chi\chi}$.

The expression within the bracket can be compared with the standard line element for the metric of the plane in polar coordinates. Therefore, this will describe a nonsingular space, if χ is a periodic variable with periodicity $2\pi(2m + a)/\sqrt{2m - a}$. So, we obtain

$$\frac{2m + a}{\sqrt{2m - a}} = R_0 \qquad (V.6).$$

We get two solutions for m

$$m_{\pm} = \frac{1}{2}\left[\frac{R_0^2}{2} - a \pm \frac{R_0}{2}\sqrt{R_0^2 - 8a}\right] \qquad (V.7)$$

Note that $R_0^2 \geq 8a$. Also, now ρ can run between $\rho_i \leq \rho < \infty$. The value of ρ_i in the two cases will be different

$$\rho_{i\pm} = \left[\frac{R_0^2}{2} - 2a \pm \frac{R_0}{2}\sqrt{R^2 - 8a}\right]^{1/2}. \qquad (V.8)$$

Unfortunately, it is not easy to arrive at a conclusion regarding the presence of negative action modes in small fluctuations around these solutions, since the perturbation equations turn out to be extremely complex. The general procedure is as follows.

Let us consider a perturbation, $h_{\mu\nu}$ of a metric $g_{\mu\nu}$ which satisfies the Euclidean vacuum field equations corresponding to Eq.(I.27)

$$G_{\mu\nu}(g_{\alpha\beta}) = \alpha\kappa\, S_{\mu\nu}(g_{\alpha\beta}), \tag{V.9}$$

where $g_{\alpha\beta}$ is with Euclidean signature. The perturbed metric $(g_{\mu\nu} + h_{\mu\nu})$ also should satisfy the field equations and so

$$G_{\mu\nu}(g_{\alpha\beta}+h_{\alpha\beta})-\alpha\kappa\, S_{\mu\nu}(g_{\alpha\beta}+h_{\alpha\beta}) = G_{\mu\nu}(g_{\alpha\beta})+\delta G_{\mu\nu}(h_{\alpha\beta})-\alpha\kappa\left[S_{\mu\nu}(g_{\alpha\beta})+\delta S_{\mu\nu}(h_{\alpha\beta})\right]$$

$$\text{So,} \qquad \delta G_{\mu\nu}(h_{\alpha\beta}) = \alpha\kappa\, \delta S_{\mu\nu}(h_{\alpha\beta}). \tag{V.10}$$

One may choose to work in a tracefree transverse gauge for $h_{\mu\nu}$, so that

$$g^{\mu\nu}h_{\mu\nu} = 0 \tag{V.11}$$

$$D_\mu h^{\mu\nu} = 0. \tag{V.12}$$

The most general traceless metric perturbation of Eq.(V.3) may be given as

$$h_{\mu\nu} = A(\rho)P^{-1}d\rho^2 - \frac{1}{3}[A(\rho)+B(\rho)]\rho^2[d\psi^2+\sin^2\psi(d\theta^2+\sin^2\theta\, d\phi^2)]+B(\rho)P\, d\chi^2. \tag{V.13}$$

This preserves both the rotational and time symmetry of the metric.

One may use the ρ–component of Eq.(V.12) to derive a relationship between A and B. The other components of this equation will be trivially satisfied. The next task is to solve the appropriate eigenvalue equation corresponding to Eq.(V.10). This equation which essentially comes out of the second variation of the action turns out to be a very complicated one involving various combinations of higher order terms [To get an idea of how it may look like, one may see Wiltshire (1988) where the Lorentzian version of the same perturbation equation has been explicitly written by choosing a de Donder gauge $(h^{\mu\nu} - \frac{1}{2}g^{\mu\nu}h^{\lambda}{}_{\lambda})_{;\mu} = 0$. The situation will not improve much in a transverse traceless gauge]. It is almost impossible to solve such an equation except probably by the use of a powerful computer symbolic manipulation program.

Since the m_+ solution approaches the 'bounce' solution given by Witten [eq.III.10] in the $\alpha \to 0$ limit, we expect a negative action mode to be present in this case. It is difficult to guess any result on the m_- solution. But it is quite possible that the actual higher order perturbation equation in each case may contain more than one negative eigenvalue.

However, as pointed out by Witten (1982), the much simpler way to see that the bounce solutions (V.3) actually describe the instability, is to perform a suitable analytical continuation of these solutions from the Euclidean to the Minkowskian space. If a real Euclidean solution remains to be a real valued Minkowski solution after the analytical continuation, it should describe the instability. This argument will also be confirmed by the fact that both the Minkowski solutions have zero energy and, therefore, represent the nonuniqueness of the assumed ground state, thus violating the positive energy theorem.

Therefore, performing the transformation $\psi \to \frac{\pi}{2} + i\tau$ on the metrics [Eq.V.3], we write the Minkowskian signature solutions as

$$ds^2 = -\rho^2 d\tau^2 + P^{-1}d\rho^2 + \rho^2 \cosh^2\tau(d\theta^2 + \sin^2\theta d\phi^2) + Pd\chi^2. \qquad (V.14)$$

As discussed in sec.III.(B), these represent the alternative spacetimes into which the assumed ground state decays. We verify below that the energy of these spacetimes is zero.

We can easily see that the presence of higher order terms will not create any problem in defining the energy integral for such a system. Because the dynamics of large distances are governed by the lower order Einstein term, the conserved energy of an asymptotically flat spacetime can be given by the usual ADM expression. For a quasi-Minkowskian spacetime, the metric $g_{\mu\nu}$ can be splitted up into its asymptotic value $\eta_{\mu\nu}$ and a deviation $h_{\mu\nu}$: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Then we can rewrite Einstein's equations as

$$G^L_{\mu\nu} = R^L_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^L = -\frac{\kappa}{2}(T_{\mu\nu} + \tau_{\mu\nu}), \qquad (V.15)$$

where the superscript L represents the linear part of the corresponding quantities. $\tau_{\mu\nu}$

includes all the terms of $G_{\mu\nu}$, nonlinear in h.

$$\begin{aligned}\tau_{\mu\nu} =& \frac{2}{\kappa}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) - R^L_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R^L \\ &+ 2\alpha[2RR_{\mu\nu} - 4R_{\mu\alpha}R^{\alpha}{}_{\nu} - 4R_{\alpha\beta}R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} + 2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} \\ &- \frac{1}{2}g_{\mu\nu}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2)]\end{aligned} \tag{V.16}$$

Then one can proceed in the standard way [Weinberg, 1972] to define the energy of such a spacetime geometry. For our purpose, we are using the energy integral defined by Deser and Soldate, 1988, for a geometry with a compactified dimension :

$$E = \frac{1}{16\pi G_5}\int_0^{2\pi R_0} dx^5 \int d^2S^k[h_{kj,j} - h_{jj,k} - h_{55,k}], \tag{V.17}$$

Now we can see that only the terms in the metric of order $1/\rho$ are relevant. However, since P contains no term of order $1/\rho$, the energy integrals for these spacetime geometries will give zero energy. These two solutions are, therefore, two counter-examples of the uniqueness of the $M^4 \times S^1$ ground state in Einstein-Gauss-Bonnet theory.

Both the solutions will have the same behaviour as that of ordinary Witten bubble, but their initial radii will be different : ρ_i+ and ρ_i-. Any of the two solutions will represent a perfectly reflecting expanding bubble of area $4\pi\rho_i^2\cosh^2\tau$ and at any time t, its radius will be $\rho(t) = \sqrt{\rho_i^2 + t^2}$. This corresponds to a distorted Minkowskian space in which the interior of the hyperboloid $x^2 - t^2 < \rho_i^2$ has been deleted [refer to Fig.9 by replacing $r \to \rho$ and $R_0 \to \rho_i$].

We can study the behaviour of the solutions in the limit $\alpha \to 0$. In that limit, two solutions of m will behave as

$$m_+ = \frac{R_0^2}{2} - \frac{3}{2}a \qquad m_- = \frac{a}{2} \tag{V.18}$$

When a=0, the solutions are given by the values of P as.

$$P_+ = 1 - \frac{R_0^2}{\rho^2} \qquad P_- = 1 \tag{V.19}$$

So, the '+' solution, in the $\alpha \rightarrow 0$ limit, approaches the Witten bubble solution. On the other hand, the second solution, in that limit, approaches the assumed vacuum state. We can say that the '+' solution is actually the modified Witten bubble solution, whereas, the '−'solution is an entirely new one, but with the same physical properties. One should note the significance of the higher order terms here. Higher order corrections not only modifies the Witten bubble solution, but also provides a new solution. Only because of the presence of nonzero string parameter (and consequently higher order terms), the second solution comes into being to be interpreted as an alternative decay solution of the Kaluza-Klein vacuum.

It is obvious from Eq.(V.8) that the initial radius ρ_i of the bubble represented by '+' solution will always be greater than or equal to $R_0/2$. But that for the '−' solution will always be less than or equal to $R_0/2$. As $\alpha \rightarrow 0$, ρ_{i-} goes as $\sqrt{a}$ and eventually becomes zero for $\alpha = 0$, which represents the Kaluza-Klein ground state.

Another interesting feature in our calculation is the relationship: $R_0^2 \geq 8a$. One can interpret this in two ways. One can say that the presence of nonzero string parameter has set a lower limit to the radius of the fifth dimension, although the upper limit is not fixed. On the other hand, as has already been pointed out, in ordinary Einstein action, the radius of the fifth dimension is undetermined. This can be determined only by including quantum corrections. Therefore, if one wishes to keep R_0 theoretically unrestricted, one can say that for any determined value of R_0, the string parameter will have an upper limit.

To probe into the geometry of these spacetimes, one may wish to study the behaviour of geodesics and scalar waves in these spacetimes. The nature of time-like and null geodesics in the Witten bubble solution was studied by Brill and Matlin (1989). We can easily see that there will not be any qualitative difference between that case and present ones, since respective 'P' in different cases is always positive and zero only at $\rho = \rho_i$. The time-like geodesics will execute oscillating behaviour in ρ, with turning points at ρ_i and at $k = \sqrt{k_\tau^2 - k_\phi^2 - k_\chi^2}$, where k_τ, k_ϕ, k_χ are constants of motion in the corresponding

directions. On the other hand, if a massless particle is directed toward the bubble, it will be reflected only once and then will move away at the speed of light.

For studying scalar waves in these spacetimes, the mathematical formalism developed in the work of Bhawal and Vishveshwara, 1990, on Witten bubble may be used here in a straightforward manner. Only the radial solutions in the present cases will differ from the radial solution given by them. But, as discussed in the appendix of their paper, appropriate coordinate transformations can bring the radial equations in the Schrodinger's form which can be studied. However, that will not provide any result qualitatively different from the Witten bubble case.

A few points discussed here will be elaborated in the next chapter.

Chapter VI

EPILOGUE

We discussed our motivation and summarized our successes and failures in various problems in the previous chapters. Here, we attempt to interrelate our failures and propose some logical extensions and improvements thereof. We indicate some important works done by others to shed some light on some unexplored prospects and problems in higher order gravity.

Let us start with black holes. One solution that holds the string of most of the works done by us is the Boulware–Deser black hole (BDBH). There is a series of other more generalised black hole solutions in the general second order Lovelock gravity [see sec.I(D)]. However, there still remains some more unsettled profound issues related to black holes in quadratic gravity.

We do not yet know the status of the singularity theorems in these theories, since the condition of the validity of these theorems may get violated by the higher order Lagrangians. Questions related to the different energy conditions (weak, strong, dominant) are also to be revisited, since they play an important role in both the classical and semi-classical aspects of various models. Since the vacuum field equations add up extra higher order terms with an indefinite sign, the above issues affect the status of the positive energy theorem which we discussed in chapter V.

One also has to explore carefully and in details the intricate issues related to the uniqueness theorem, quantum coherence problem arising out of the evaporation of the black holes and the back reaction problem in this context.

Another important problem that remains to be unsolved is the investigation of the classical stability (gravitational) of these black holes. This can be checked by performing similar analyses done in the case of four dimensional Lorentzian Schwarzschild solution by

several authors [Regge and Wheeler, 1957; Vishveshwara, 1970; Edelstein and Vishveshwara, 1970]. The actual calculation, however, faces severe problems by the fact that the perturbation equations are extremely complex. One needs to develop a powerful computer symbolic manipulation program to solve these problems. It has, however, been argued by Boulware and Deser (1985) that the asymptotically de Sitter branch of these solutions ($M > 0$, $\alpha > 0$) is unstable.

The same problem has thwarted our attempt to find negative modes in small oscillations around the 'bounce' solutions obtained by us in chapter V, which may interpolate between the $M^4 \times S^1$ ground state and alternative zero energy bubble solutions in Einstein-Gauss-Bonnet (EGB) theory.

One may guess from the structure of the higher order perturbation equation that there may exist some possibilty of obtaining more than one negative eigenvalue. In quantum field theory, it was pointed out and proved by Coleman (1988) that, in all cases of the decay of a metastable state by quantum tunneling, the second variation derivative of the Euclidean action at the bounce has one and only one negative eigenvalue. However, the same problem has not yet been investigated in the case of an Euclidean Lagrangian with higher order or higher derivative terms.

The 'bounce' solution obtained by Witten (1982) representing the decay of $M^4 \times S^1$ in ordinary GR possesses only one negative eigenvalue in the functional determinant for small oscillations around it. This result actually stems out from the fact that there also exist only one negative eigenvalue in the Lichnerowitz equation representing the perturbation of the Euclidean Schwarzschild solution which Witten (1982) used.

So, in our context, it will be very interesting to know whether the perturbation around the BDBH solution possesses more than one negative eigenvalue. The comparison of this result with that in a higher order or higher derivative extension of the field theoretic analysis by Coleman (1988) may give us good insight into the nature of the vacuum decay in these theories. The implication of the extra negative modes may then be studied.

In this context, we would also like to point out a related problem which has not attracted much attention. Although the study of scalar perturbations in higher dimensional spacetimes is relatively straightforward the case of 'higher spin' perturbations is technically more complicated and should prove to be of mathematical interest at least.

Now, we would like to mention two significant related works done by others, which may have far-reaching consequences on the future of Lovelock gravity.

An interesting study related to the classical stability of the EGB theory and, in general, the Lovelock gravity with compactified higher dimensions has been reported by Sokolowski *et al* (1991). They showed that the presence of dilaton (and of other geometric scalar fields) may render the possible ground state solutions of the reduced theory (i.e. the Minkowski and anti-de-Sitter spaces) unstable against perturbation of the scalar field. They have argued that the Gauss-Bonnet combination should, therefore, be discarded because this poses a serious problem which, unlike in the case of higher dimensional Einstein gravity, cannot be removed by field redefinitions.

Another interesting and somewhat controversial point has been raised in a series of papers by Simon (1990,91,92). He pointed out that if the second order terms are thought to be semiclassical perturbation corrections (of order $\hbar$), some new nonperturbative solutions may arise in this theory [note that one of our 'bounce' solutions. m_- in Eq.V.7 falls in this category]. But unlike the effective action and the field equations which generate them, most of these new solutions do not satisfy the initial perturbative ansatz, i.e. they are not perturbatively expandable in $\hbar$. The anti-de-Sitter branch of the solutions given by Boulware and Deser (1985) has this property.

Simon's argument is that the most self-consistent approach would be to discard these nonperturbative solutions because semiclassical gravity is only expected to approximate a perturbative expansion of the full theory and so, these 'pseudo-solutions' will fail to give any insight into the nonperturbative features of the full theory of quantum gravity. By these arguments, he showed that flat space is perturbatively stable to first order in

$\hbar$ against quantum fluctuations in semiclassical approximations to quantum gravity, although the past predictions had gone to the contrary [Hartle and Horowitz, 1981]. Similar arguments rule out Starobinsky (1980) inflation (de Sitter solutions driven only by higher order curvature terms).

All these points are to be cautiously studied before we arrive at any final conclusion. We would like to point out, while drawing an end to this thesis, that our physical intuition always suffers a drawback gaining its experience mostly from the ordinary theories. Higher order or higher derivative theories have never been extensively studied despite the fact that these may naturally arise in different branches of Physics [e.g., the relativistic model of the classsical radiating electron given by Dirac(1938)]. We should be careful enough before making any statement or jumping into any conclusion regarding any problem in these theories.

The not–so–happy marriage of gravity with quantum theory gave a 'natural' birth to the twins— higher dimensional and higher order gravity. They are here to stay and grow up and only God knows, when they will find themselves at the limiting end of the complete thory of quantum gravity and will come to know what God only knows.

APPENDIX A

Deflection of Null Ray in Higher Dimensional Black Hole Spacetime

This is a simple calculation based on standard procedure that we (Bhawal B. and Mani H.S., 1988, unpublished) did to arrive at an expression for the measure of the deflection of null ray propagating in higher dimensional black hole spacetime. We found that, under certain assumptions, the general expressions obtained for even and odd number of dimensions differ from each other.

In any static, spherically symmetric higher dimensional spacetime of the form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega_n^2, \tag{A.1}$$

where $d\Omega_n^2$ is given by Eq.(II.3), one may consider the orbit of the null ray to be in the equatorial plane (i.e. where all polar angles $\theta_i = \pi/2$ for $i = 2, 3, \ldots, n$), since the field is isotropic.

Then proceeding in the standard way [Weinberg, 1972], one arrives at the following expression for the deflection of null ray by the gravitational field.

$$\Delta\theta_1 = 2|T| - \pi \tag{A.2}$$

$$T = \int_r^\infty A^{1/2}(r)\left[\left(\frac{r}{b}\right)^2 \frac{B(b)}{B(r)} - 1\right]^{-1/2} \frac{dr}{r} \tag{A.3}$$

where b is the distance of the closest approach to the central point.

Let us now choose the background metric to be the higher dimensional Schwarzschild-de Sitter spacetime [Dianyan, 1988] given by Eq.(A.1) and

$$B(r) = A^{-1}(r) = 1 - \frac{r_0^m}{r^m} - \frac{2\Lambda r^2}{(m+1)(m+2)}, \qquad r_0^m = 2GM \tag{A.4}$$

Λ is the cosmological constant. $m = D - 3$.

Integrating T for the above expressions of A and B is very difficult. We assume that throughout the orbit of the null ray, r (or, b) is much greater than r_0. Then, since Λ is

also a very small quantity, one may write

$$A(r) = B^{-1}(r) = 1 + \frac{r_0^m}{r^m} + \frac{2\Lambda r^2}{(m+1)(m+2)} \tag{A.5}$$

Then T may be calculated to be

$$T \simeq \int_b^\infty \frac{dr}{r}\left[\frac{r^2}{b^2} - 1\right]^{-1/2}\left[1 + \frac{r_0^m}{2r^m} + \frac{r_0^m}{2r^{m-2}b^m}\frac{b^{m-1} + rb^{m-2} + \cdots + b^{m-1}}{r+b} \right.$$
$$\left. - \frac{2\Lambda r^2}{(m+1)(m+2)}\right] \tag{A.6}$$

Making change of variable to $x = b/r$, we obtain the deflection of the null ray to be

$$\Delta\theta_1 = \frac{r_0^m}{b^m}\int_0^1 \frac{dx}{(1-x^2)^{1/2}}\left[x^m + \frac{1 + x + \cdots + x^{m-1}}{1+x}\right]$$
$$- \int_0^1 \frac{dx}{x^2(1-x^2)^{1/2}}\frac{4\Lambda b^2}{(m+1)(m+2)} \tag{A.7}$$

The last term containing Λ blows up. Therefore, from now onwards we set $\Lambda = 0$ or equivalently, we confine our discussion to ordinary higher dimensional Schwarzschild spacetime.

For $\Lambda = 0$,

$$\Delta\theta_1 = \frac{r_0^m}{b^m}\int_0^1 \frac{dx}{\sqrt{1-x^2}}\left[\frac{1 + x + \cdots + x^{m+1}}{1+x}\right] \tag{A.8}$$

When D is odd (or, $m+1$ is odd)

$$1 + x + \cdots + x^{m+1} = (1+x)(1 + x^2 + x^4 + \cdots x^m). \tag{A.9}$$

When D is even (or, $m+1$ is even)

$$1 + x + \cdots + x^{m+1} = 1 + (1+x)(x + x^3 + x^5 + \cdots x^m). \tag{A.10}$$

So, when D is odd, deflection of null ray is given by

$$\Delta\theta_{1O} = \frac{r_0^m}{b^m}\int_0^1 \frac{dx}{\sqrt{1-x^2}}\left[1 + x^2 + x^4 + \cdots + x^m\right] \tag{A.11}$$

The corresponding expression for D even is given by

$$\Delta\theta_{1E} = \frac{r_0^m}{b^m}\int_0^1 \frac{dx}{\sqrt{1-x^2}}\left[\frac{1}{1+x} + x + x^3 + x^5 + \cdots + x^m\right]. \tag{A.12}$$

All the integrals appearing in the above expressions can be written in terms of Beta or Gamma functions as shown below. For any value of $m = p$ (say)

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}x^p = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p}{2}+1\right)}. \tag{A.13}$$

The following expressions can, therefore, be obtained

$$\Delta\theta_{1O} = \frac{\pi GM}{b^m}\left[1 + \sum_{k=1}^{m/2}\frac{(2k-1)(2k-3)\cdots 3.1}{2^k k!}\right] \tag{A.14}$$

$$\Delta\theta_{1E} = \frac{4GM}{b^m} + \frac{GM}{b^m}\sum_{k=1}^{(m-1)/2}\frac{2^{k+1}k!}{(2k+1)(2k-1)\cdots 5.3.1}. \tag{A.15}$$

The expression of $\Delta\theta_{1E}$ for $D = 4$ tallies with the standard expression obtained in four dimensional analysis ($= 4GM/b$).

If one wants to compare the values of deflections in two Schwarzschild spacetimes of different dimensions, one may do so by choosing units $c = G = 1$ and dimensionless variable for the radial coordinate. It is not easy to arrive at any general conclusion since the result crucially depends on the value of b as well as the corresponding number of dimensions.

APPENDIX B

Alternative Scalar Wave Solutions in The Witten Bubble Background

Here, we present alternative forms of solutions of both the radial and τ-equations, obtained by different procedures, for the Klein-Gordon equation in the Witten Bubble spacetime. Since these solutions are not convenient for formulating scattering and other problems discussed in chapter IV, we have not used them in our work. However, for the sake of completeness and for possible use elsewhere, we describe these here.

(i) τ-*Equation*

The τ-part of the separated Klein-Gordon equation, which we will denote here as T, instead of $T^{\ell}_{i\omega}$, satisfies Eq.(IV.7).

Let us do the coordinate transformation

$$\tau \longrightarrow i\left(\frac{\pi}{2} - \tau'\right) \tag{B.1}$$

This is equivalent to the Euclidean continuation of Eq.(IV.7). Then introducing the variable

$$p = \cos\tau', \tag{B.2}$$

we can write Eq.(IV.7) as

$$(1-p^2)\frac{d^2T^E}{dp^2} - 3p\frac{dT^E}{dp} + \left(\alpha - \frac{\ell(\ell+1)}{1-p^2}\right)T^E = 0 \tag{B.3}$$

where by T^E, we represent the Euclidean continuation of function T. Also,

$$\alpha = -\omega^2 - 1. \tag{B.4}$$

$$\text{Defining} \qquad Z = (1-p^2)^{-\ell/2}T^E \tag{B.5}$$

we get

$$(1-p^2)\frac{d^2Z}{dp^2} - p(2\ell+3)\frac{dZ}{dp} + [k(k+2) - \ell(\ell+2)]Z = 0 \tag{B.6}$$

where we have chosen

$$\alpha = k(k+2). \tag{B.7}$$

Equation(B.6) can now be written as

$$(1-p^2)\frac{d^2Z}{dp^2} - p(2\mu+1)\frac{dZ}{dp} + \lambda(\lambda+2\mu)Z = 0 \tag{B.8}$$

by defining

$$\begin{aligned} \mu =& \ell + 1 \\ \lambda =& k - \ell. \end{aligned} \tag{B.9}$$

This is the standard Gegenbauer equation, which has two solutions expressed in terms of hypergeometric series.

$$C_\lambda^\mu(p) = \frac{\Gamma(2\mu+\lambda)}{\Gamma(\lambda+1)\Gamma(2\mu)} F\big(-\lambda, \lambda+2\mu; \mu+\frac{1}{2}; \frac{1-p}{2}\big) \tag{B.10}$$

$$D_\lambda^\mu(p) = 2^{-1-\lambda}\frac{\Gamma(\mu)\Gamma(2\mu+\lambda)}{\Gamma(\mu+\lambda+1)} F\big(\mu+\frac{1}{2}\lambda, \mu+\frac{\lambda}{2}+\frac{1}{2}; \mu+\lambda+1; p^2\big) \tag{B.11}$$

Therefore, using Eq.(B.5), we get two solutions for Eq.(B.3)

$$T_1^E = (\sin\tau')^\ell C_\lambda^\mu(\cos\tau') \tag{B.12}$$

$$T_2^E = (\sin\tau')^\ell D_\lambda^\mu(\cos\tau') \tag{B.13}$$

Performing the reverse transformation of Eq.(B.1), or equivalently, continuing back to the Minkowski solutions, we obtain

$$T_1 = (\cosh\tau)^\ell C_\lambda^\mu(-i\sinh\tau) \tag{B.14}$$

$$T_2 = (\cosh\tau)^\ell D_\lambda^\mu(-i\sinh\tau) \tag{B.15}$$

(ii) *Radial Equation*

We can get a Frobenius series solution of Eq.(IV.14), if we assume it first to be of the form

$$\mathcal{R} \sim (r-R_0)^q \sum_{n=0}^{\infty} a_n (r-R_0)^n.$$

Then substituting this in Eq.(IV.14), we obtain the following equation

$$
\begin{aligned}
0 =& q(q-1)z^{q-2}\sum_0^\infty a_n z^n + 2qz^{q-1}\sum_1^\infty n a_n z^{n-1} + z^q \sum_2^\infty n(n-1)a_n z^{n-2} \\
&+ \left(1+\frac{3z}{2R_0}+\frac{5z^2}{4R_0^2}+\frac{27z^3}{8R_0^3}+\cdots\right)\left[qz^{q-2}\sum_0^\infty a_n z^n + z^{q-1}\sum_1^\infty n a_n z^{n-1}\right] \qquad (B.16)\\
&+ \frac{\omega^2+1}{2R_0}\sum_0^\infty \left(-\frac{z}{2R_0}\right)^n z^{q-1}\sum_0^\infty a_n z^n,
\end{aligned}
$$

where $z = r - R_0$.

Equating the coefficients of different powers of z, we obtain $q = 0$ and can determine different a_n, so that the solution turns out to be

$$\mathcal{R}_1 = a_0\left[1 - \frac{\omega^2+1}{2R_0}(r-R_0) + \frac{\omega^4+6\omega^2+5}{16R_0^2}(r-R_0)^2 - \cdots\right]. \qquad (B.17)$$

A second solution can be found to be of the form

$$\mathcal{R}_2 = \ln(r-R_0)\sum_0^\infty a_n(r-R_0)^2 + \sum_0^\infty b_n(r-R_0)^n \qquad (B.18)$$

where a_n, b_n are constants to be determined from Eq.(B.15).

The first solution behaves properly throughout the range of the variable r, whereas, the second solution blows up at $r = R_0$.

APPENDIX C

A Special Kind of Coordinate Transformation

Coordinate transformations like Eq.(IV.15) are widely used in many situations both in flat and curved spacetimes to bring the radial equation to the Schrödinger form, e.g. the 'tortoise' coordinates in the Schwarzschild spacetime. However, these were considered to be just some mathematical operation. Their actual significance does not seem to have been discussed in the literature. Here, we will attempt to give a general basis for this.

For a metric in which the Klein-Gordon equation is separable and g_{00}, g_{rr} are independent of time, one can always obtain the radial equation in Schrödinger form just by choosing a null coordinate system.

In a static spacetime, if one solves the Klein-Gordon equation for a massive scalar field, one obtains the following eigenvalue equation after separating out the temporal part which will be of the form $e^{+i\omega t}$,

$$\frac{1}{\sqrt{-g}} g_{00} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) + g_{00} m^2 \Phi = \omega^2 \Phi \tag{C.1}$$

Now, if it is a two dimensional metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2, \tag{C.2}$$

let us try to get a null-vector η_i by introducing a new coordinate r^*,

$$\eta^i = (1, \frac{dr}{dr^*})$$

so that,

$$\eta^i \eta_i = -A + (\frac{dr}{dr^*})^2 B = 0$$

$$\text{or,} \quad \frac{dr^*}{dr} = \sqrt{\frac{B}{A}}. \tag{C.3}$$

$$\text{Then} \quad ds^2 = A(-dt^2 + dr^{*2})$$

and Eq.(C.1) becomes

$$-\frac{d^2\Phi}{dr^{*2}} + m^2 A\Phi = \omega^2\Phi \tag{C.4}$$

Only the mass term contributes to the effective potential. For $m = 0$, this is just a free wave solution.

In a general dimensional spacetime, if $g_{\alpha\alpha}$, where $\alpha \neq t, r$ be r-dependent, then there will be an extra first derivative term in Eq.(C.4). This first derivative term can be easily eliminated by suitably defining a new radial function and the Schrödinger equation can be obtained. The r-dependence of $g_{\alpha\alpha}$ will actually contribute to the effective potential of this equation.

If $g_{\alpha\alpha}$ is also time-dependent, the eigenvalue equation of Φ will not be of the form (C.1). But one can easily see that this will not create any problem in getting a Schrödinger form by choosing a null coordinate.

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